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Differential forms via the Bernstein–Gelfand–Gelfand resolution for quantized irreducible flag manifolds

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Dedicated to the memory of Leonid L. Vaksman (1951–2007)

Abstract

The quantum group version of the Bernstein–Gelfand–Gelfand resolution is used to construct a double complex of $U_q(\mathfrak{g})$-modules with exact rows and columns. The locally finite dual of its total complex is identified with the de Rham complex for quantized irreducible flag manifolds.

MSC: 17B37; 58B32

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1. Introduction

Over the last two decades a vast amount of papers have been devoted to the translation of classical geometric concepts to coordinate algebras appearing in the theory of quantum groups. It is a recurring theme that such constructions are possible if the underlying geometric object can be expressed in purely Lie algebraic terms. A list of examples where this translation has a very simple and compelling form might include the standard definition of the q-deformed coordinate algebra $\mathbb{C}_q[G]$ inside the dual Hopf algebra of $U_q(\mathfrak{g})$ [9, 9.1.1] or the construction of the quantum group version of the homogeneous coordinate ring of a flag manifold [9, 9.1.6]. Certainly, one always aims for quantum effects, as for instance Drinfeld duality, which transcend the classical undeformed situation. However, we will not encounter significant quantum effects in this paper.

Differential forms are an example of a geometric concept where the translation from the classical to the quantum group setting is far from obvious in general. However, there is a notion of covariant differential calculus on quantum spaces, introduced by S.L. Woronowicz [24], which has attracted much attention for many years ([12] and references therein). It soon turned out that for a general quantum space there exists no canonical construction of a covariant

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differential calculus. However, in [7] we showed that for quantized irreducible flag manifolds G/P_S where G is a simple complex affine algebraic group and P_S a standard parabolic subgroup there exists a q-analogue of the de Rham complex which in many respects behaves like its undeformed counterpart. The aim of the present paper is to relate this complex to its Lie algebraic shadow, the Bernstein–Gelfand–Gelfand (BGG) resolution. In the quantum case such a construction was suggested by L.L. Vaksman and a first indication of its feasibility can be found in [22] where generalized Verma modules are used to obtain q-analogues of differential one forms.

The main result of the present paper, Theorem 7.14, states that the de Rham complex investigated in [7] can also be obtained as the locally finite dual of a BGG-like sequence of $U_q(\mathfrak{g})$-modules induced by $U_q(\mathfrak{l}_S)$-modules, where $\mathfrak{l}_S$ denotes the Levi factor of the parabolic subalgebra $\mathfrak{p}_S \subset \mathfrak{g}$. More precisely the BGG resolution for quantum groups [4] is used to define quantum analogues of the complexes of holomorphic and antiholomorphic differential forms on flag manifolds (Proposition 7.8 and Section 7.3). In Section 7.4 we introduce a double complex the rows and columns of which are closely related to the BGG resolutions used to obtain the holomorphic and antiholomorphic differentials, respectively. The desired de Rham complex is then obtained as the locally finite dual of the total complex of this double complex.

The reason why we have to consider $U_q(\mathfrak{l}_S)$-modules, instead of $U_q(\mathfrak{p}_S)$-modules as one might expect, lies in the definition of the coordinate algebra $\mathbb{C}_q[G/L_S]$ describing the quantum flag manifold. Its classical counterpart is the coordinate ring of the affine algebraic variety G/L_S where L_S denotes the Levi factor of P_S. The advantage of this approach lies in the fact that $\mathbb{C}_q[G]$ is a Hopf–Galois extension of $\mathbb{C}_q[G/L_S]$. Thus M. Takeuchi's categorical equivalence [23] applies and one can make use of results on differential calculi on quantum homogeneous spaces [5].

A result similar in spirit has recently been obtained in [21]. In that paper the universal higher order differential calculus constructed in [22] is identified with the category $\mathcal{O}$ dual of the q-version of the BGG resolution. Hence in the approach taken in [21] Takeuchi's categorical equivalence is not available and the authors have to revert to specialization techniques. In the present paper, on the other hand, all results are proved for any deformation parameter $q \in \mathbb{C}$ which is not a root of unity.

As the reader might at first be put off by the technical nature of our paper we now state the main result in the special case of one-dimensional quantum complex projective space also known as standard quantum sphere. This simplest example of an irreducible quantized flag manifold in itself has been subject to various publications, e.g. [3,20,15]. We believe that our analysis will lead to new insight even in this simplest case.

Recall that $U_q(\mathfrak{sl}_2)$ denotes a Hopf algebra generated by elements E, F, K, and K^{-1} and relations given for instance in [12, 3.1]. Let $U_q(\mathfrak{l})$ denote the subalgebra generated by K and K^{-1}, for $n \in \mathbb{Z}$ let $V(n)$ denote the one-dimensional $U_q(\mathfrak{l})$-module generated by one element v_n with the action $Kv_n = q^{2n}v_n$, and define $W(n,m) := U_q(\mathfrak{sl}_2) \otimes_{U_q(\mathfrak{l})} V(n-m)$. Note that $W(0,0)$ is a coalgebra and that $W(n,m)$ is a left and right comodule over $W(0,0)$ with coactions given by

$$u \otimes v_n \mapsto (u_{(1)} \otimes v_0) \otimes (u_{(2)} \otimes v_n) \quad \text{and} \quad u \otimes v_n \mapsto (u_{(1)} \otimes v_n) \otimes (u_{(2)} \otimes v_0),$$

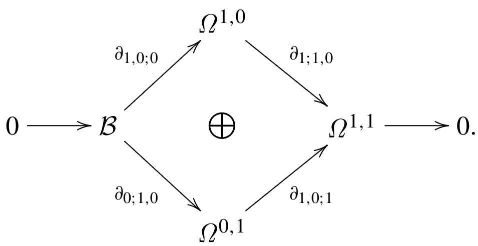
respectively, where Sweedler notation is used. Consider the following sequence of $U_q(\mathfrak{sl}_2)$-modules, $W(0,0)$-bicomodules

$$0 \longrightarrow W(1,1) \xrightarrow{\begin{pmatrix} -\varphi_{1;1,0} \\ \varphi_{1,0;1} \end{pmatrix}} \begin{matrix} W(1,0) \\ \oplus \\ W(0,1) \end{matrix} \xrightarrow{(\varphi_{1,0;0},\ \varphi_{0;1,0})} W(0,0) \longrightarrow 0 \tag{1}$$

where $\varphi_{a;b,c}(u \otimes v_{a-b}) = uF \otimes v_{a-c}$ and $\varphi_{a,b;c}(u \otimes v_{a-c}) = uE \otimes v_{b-c}$. The locally finite dual of $W(a,b)$ is defined by

$$\Omega^{a,b} = \{ f \in W(a,b)^* \mid \dim(fU_q(\mathfrak{sl}_2)) < \infty \}$$

where $W(a,b)^*$ denotes the linear dual space of $W(a,b)$. As $W(0,0)$ is a $U_q(\mathfrak{sl}_2)$-module coalgebra the space $\mathcal{B} = \Omega^{0,0}$ is a $U_q(\mathfrak{sl}_2)$-module algebra and as such $\mathcal{B}$ coincides with the standard quantum sphere. Dualizing (1) one obtains a sequence of $U_q(\mathfrak{sl}_2)$-module $\mathcal{B}$-bimodules



The main result of this paper, Theorem 7.14, states in this special case, that this sequence coincides with the well known de Rham complex [18] over the standard quantum sphere $\mathcal{B}$. As an application one can for instance immediately read off the twisted cyclic cocycle calculated in [20, Lemma 4.4].

We now describe the contents of each section of this paper in some detail. In Section 2 we fix notation. Moreover, we compare the standard resolution of the trivial module with the parabolic version of the BGG resolution and show that these two coincide if $\mathfrak{g}/\mathfrak{p}_S$ is irreducible. This result should be well known but we were not able to track it in the literature. On the one hand it explains once again why it is necessary to assume irreducibility of the considered flag manifolds. On the other hand, it implies that certain weights $w.0$ are incomparable in the Bruhat order.

Section 3 serves purely to fix notation for quantum groups and to recall M.S. Kébé's results on triangular decompositions of $U_q(\mathfrak{g})$ with respect to parabolic subalgebras. In Section 4 we quickly review the q-analogue of the BGG resolution which by [4] is exact if q is not a root of unity. Section 5 is devoted to $U_q(\mathfrak{g})$-modules induced by irreducible $U_q(\mathfrak{l}_S)$-modules. We denote the category of finite direct sums of such modules by $\mathcal{W}$. In Section 5.3 we derive technical properties of standard maps between objects in $\mathcal{W}$ related to the BGG resolution. The locally finite duals of objects in $\mathcal{W}$ are interpreted as yet another realization of Takeuchi's categorical equivalence in Section 6.

The main technical work is done in the final Section 7. First the main results from [7] are recalled. Then the differential calculi $(\Gamma_\partial, \partial)$, $(\Gamma_{\overline{\partial}}, \overline{\partial})$, and $(\Gamma_\mathrm{d}, \mathrm{d})$ from that paper are interpreted as locally finite duals of BGG-like sequences in $\mathcal{W}$.

Explicit calculations flooded by symbols are inherent to proofs in quantum group theory. For the convenience of the reader we have collected all commonly used notation in order of appearance in an Appendix.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the positive integers, the integers, and the complex numbers, respectively. We write $\mathbb{N}_0$ to denote the nonnegative integers.

2.1. Notation

First, to fix notation some general notions related to Lie algebras are recalled. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra of rank r and let $\mathfrak{h} \subset \mathfrak{g}$ be a fixed Cartan subalgebra. Let $R \subset \mathfrak{h}^*$ denote the root system associated with $(\mathfrak{g}, \mathfrak{h})$. Choose an ordered basis $\pi = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots for R and let R^+ (resp. R^-) be the set of positive (resp. negative) roots with respect to π. Moreover, let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the corresponding triangular decomposition. Identify $\mathfrak{h}$ with its dual via the Killing form. The induced nondegenerate symmetric bilinear form on $\mathfrak{h}^*$ is denoted by $(\cdot,\cdot)$. The root lattice $Q = \mathbb{Z}R$ is contained in the weight lattice $P = \{\lambda \in \mathfrak{h}^* \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z}\, \forall \alpha_i \in \pi\}$ where $\alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i)$. In order to avoid roots of the deformation parameter q in the following sections we rescale $(\cdot,\cdot)$ such that $(\cdot,\cdot) : P \times P \to \mathbb{Z}$. For $\mu, \nu \in P$ we write $\mu \geq \nu$ if $\mu - \nu$ is a sum of positive roots. The height function $\mathrm{ht} : Q \to \mathbb{Z}$ is defined by $\mathrm{ht}(\sum_{i=1}^r n_i \alpha_i) = \sum_{i=1}^r n_i$.

The fundamental weights $\omega_i \in \mathfrak{h}^*$, $i = 1, \ldots, r$, are characterized by $(\omega_i, \alpha_j^\vee) = \delta_{ij}$. Let P^+ denote the set of dominant integral weights, i. e. the $\mathbb{N}_0$-span of $\{\omega_i \mid i = 1, \ldots, r\}$. Recall that $(a_{ij}) := (2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))$ is the Cartan matrix of $\mathfrak{g}$ with respect to π. We will write $Q^+ = \mathbb{N}_0 R^+$.

For $\mu \in P^+$ let $V(\mu)$ denote the finite dimensional irreducible $\mathfrak{g}$-module of highest weight μ. Moreover, let $\Pi(V(\mu))$ denote the set of weights of the $\mathfrak{g}$-module $V(\mu)$.

Let G denote the connected simply connected complex Lie group with Lie algebra $\mathfrak{g}$. For any set $S \subset \pi$ of simple roots define $Q_S = \mathbb{Z}S$, $Q_S^+ = Q_S \cap Q^+$, and $R_S^\pm := Q_S \cap R^\pm$. Let P_S and P_S^{op} denote the corresponding standard parabolic subgroups of G with Lie algebra

$$\mathfrak{p}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+ \cup R_S^-} \mathfrak{g}_\alpha, \qquad \mathfrak{p}_S^{\mathrm{op}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^- \cup R_S^+} \mathfrak{g}_\alpha, \tag{2}$$

respectively. Moreover,

$$\mathfrak{l}_S := \mathfrak{h} \oplus \bigoplus_{\alpha \in R_S^+ \cup R_S^-} \mathfrak{g}_\alpha$$

is the Levi factor of $\mathfrak{p}_S$ and $L_S = P_S \cap P_S^{\mathrm{op}} \subset G$ denotes the corresponding subgroup.

The generalized flag manifold G/P_S is called irreducible if the adjoint representation of $\mathfrak{p}_S$ on $\mathfrak{g}/\mathfrak{p}_S$ is irreducible. Equivalently, $S = \pi \setminus \{\alpha_i\}$ where α_i appears in any positive root with coefficient at most one. For a complete list of all irreducible flag manifolds consult e.g. [1, p. 27]. Note that the irreducible flag manifolds coincide with the irreducible compact Hermitian symmetric spaces [8, Section X.6.3]

Define $P_S^+ := \{\lambda \in P \mid (\lambda, \alpha_i)/d_i \in \mathbb{N}_0 \, \forall \alpha_i \in S\}$. To $\lambda \in P_S^+$ we associate the finite dimensional, irreducible $\mathfrak{l}_S$-module $M(\lambda)$ of highest weight λ.

Let W denote the Weyl group of $\mathfrak{g}$ generated by the reflections corresponding to the simple roots in π. For any $\alpha \in R^+$ let $s_\alpha \in W$ denote the reflection on the hyperplane orthogonal to α with respect to $(\cdot,\cdot)$. Let $W_S \subset W$ denote the subgroup generated by the reflections corresponding to simple roots in S. Moreover, define

$$W^S = \{w \in W \mid R_S^+ \subset wR^+\}.$$

By a well known result of B. Kostant any element $w \in W$ can be decomposed uniquely in the form $w = w_S w^S$ where $w_S \in W_S$ and $w^S \in W^S$. Moreover, if l denotes the length function on W then this decomposition satisfies $l(w) = l(w_S) + l(w^S)$.

The following technical lemma will be used in the proof of Propositions 6.1 and 6.5.

Lemma 2.1. *For any $\lambda \in P_S^+ \cap P$ there exist $\mu \in P^+$ which allows an injective $\mathfrak{l}_S$-module map $M(\lambda) \hookrightarrow V(\mu)$.*

Proof. Choose $w \in W$ such that $\mu := w^{-1}\lambda \in P^+$. Write $w = w_S w^S$ where $w_S \in W_S$ and $w^S \in W^S$. Then $w^S\mu = w_S^{-1}\lambda$. Let $v_{w^S\mu} \in V(\mu)$ denote a nonzero vector of weight $w^S\mu$. Note that $v_{w^S\mu}$ is a highest weight vector for $\mathfrak{l}_S$. Indeed, if $w^S\mu + \alpha_i \in \Pi(V(\mu))$ for some $\alpha_i \in S$ then $(w^S)^{-1}(w^S\mu + \alpha_i) = \mu + (w^S)^{-1}\alpha_i \notin \Pi(V(\mu))$ since $(w^S)^{-1}\alpha_i \in R^+$. Therefore $w_S^{-1}\lambda \in P_S^+$ and $\lambda \in P_S^+$ and hence $w^S\mu = w_S^{-1}\lambda = \lambda$. ■

Recall that the shifted action of the Weyl group W on P is defined in terms of the ordinary Weyl group action by

$$w.\mu = w(\mu + \rho) - \rho$$

where ρ is half the sum of all positive roots or equivalently $\rho = \sum_{i=1}^r \omega_i$. Moreover, for $w, w' \in W$ write $w \to w'$ if there exists $\alpha \in R^+$ such that $w = s_\alpha w'$ and $l(w) = l(w') + 1$. The Bruhat order $\leq$ on W is then given by the relation

$$w \leq w' \Leftrightarrow \text{there exist } n \geq 1 \text{ and } w_2, \ldots, w_{n-1} \in W \text{ such that } w = w_1 \to w_2 \to \cdots \to w_n - w'.$$

2.2. Standard resolution and BGG resolution

Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{p}$ a subalgebra. In [2] I.N. Bernstein, I.M. Gelfand, and S.I. Gelfand have given the following generalization of the standard resolution of Lie algebra cohomology. The adjoint action of $\mathfrak{p}$ on $\mathfrak{g}/\mathfrak{p}$ endows each exterior product $\Lambda^k(\mathfrak{g}/\mathfrak{p})$ with the structure of a $U(\mathfrak{p})$-module. Define

$$D_k = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \Lambda^k(\mathfrak{g}/\mathfrak{p})$$

and

$$d_0 : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C} = D_0 \to \mathbb{C}, \qquad u \otimes x \mapsto \varepsilon(u)x$$

where ε denotes the counit of $U(\mathfrak{g})$. Moreover, for $k \geq 1$ define operators $d_k : D_k \to D_{k-1}$ in the following way. Let $X_1, \ldots, X_k$ be elements of $\mathfrak{g}/\mathfrak{p}$. Let $Y_1, \ldots, Y_k \in \mathfrak{g}$ be arbitrary representatives of $X_1, \ldots, X_k$, respectively, and put

$$d_k(X \otimes X_1 \wedge \cdots \wedge X_k) = \sum_{i=1}^{k} (-1)^{i+1}(XY_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k) + \sum_{1 \leq i < j \leq k} (-1)^{i-j}(X \otimes \overline{[Y_i, Y_j]} \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_k). \tag{3}$$

Here $X \in U(\mathfrak{g})$ and we write $\overline{Y}$ for the image of the element $Y \in \mathfrak{g}$ in $\mathfrak{g}/\mathfrak{p}$. Moreover, $\hat{X}$ denotes omission of the element X. One obtains a complex

$$D_* : 0 \leftarrow \mathbb{C} \overset{d_0}{\leftarrow} D_0 \overset{d_1}{\leftarrow} D_1 \overset{d_2}{\leftarrow} D_2 \overset{d_3}{\leftarrow} \ldots$$

which is exact by [2, Thm. 9.1]. In general the complex D_* does not have an analogue for quantum universal enveloping algebras.

Let now $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra and $\mathfrak{p}_S \subset \mathfrak{g}$ a standard parabolic subalgebra as in the previous subsection.

For any irreducible highest weight module $V(\mu)$ of $\mathfrak{g}$, where $\mu \in P^+$, in generalization of [2] J. Lepowsky [14] constructed an exact sequence of $U(\mathfrak{g})$-modules

$$0 \leftarrow V(\mu) \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_{\dim(\mathfrak{g}/\mathfrak{p}_S)} \leftarrow 0$$

where

$$C_n = \bigoplus_{\substack{w \in W^S, \\ l(w)=n}} U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_S)} M(w.\mu).$$

Here the differentials are given as linear combinations of standard maps of the occurring generalized Verma modules. In particular if $\mu = 0$ one obtains an exact sequence

$$C_* : 0 \leftarrow \mathbb{C} \overset{\delta_0}{\leftarrow} C_0 \overset{\delta_1}{\leftarrow} C_1 \overset{\delta_2}{\leftarrow} \cdots \leftarrow C_{\dim(\mathfrak{g}/\mathfrak{p}_S)} \leftarrow 0. \tag{4}$$

For general parabolics the sequences of $U(\mathfrak{g})$-modules C_* and D_* are not isomorphic. Indeed, if $\mathfrak{g}/\mathfrak{p}_S$ is not irreducible then not even D_1 and C_1 need to be isomorphic. However, one has the following result.

Proposition 2.2. *If $\mathfrak{g}/\mathfrak{p}_S$ is irreducible then the corresponding complexes of $U(\mathfrak{g})$-modules (C_*, δ_*) and (D_*, d_*) are isomorphic.*

Proof. Consider the Lie subalgebra

$$\mathfrak{u}_S^- = \bigoplus_{\alpha \in R^- \setminus R_S^-} \mathfrak{g}_\alpha \subset \mathfrak{g}.$$

One has decompositions $\mathfrak{g} = \mathfrak{u}_S^- \oplus \mathfrak{p}_S$ and $U(\mathfrak{g}) \cong U(\mathfrak{u}_S^-) \otimes U(\mathfrak{p}_S)$ by the Poincaré–Birkhoff–Witt Theorem. Note that both D_* and C_* are free resolutions of the trivial left $U(\mathfrak{u}_S^-)$-module $\mathbb{C}$. Thus both sequences can be used to compute $\mathrm{Tor}_j^{U(\mathfrak{u}_S^-)}(\mathbb{C}, \mathbb{C})$ where the first entry $\mathbb{C}$ denotes the trivial right $U(\mathfrak{u}_S^-)$-module. Note that if $\mathfrak{g}/\mathfrak{p}_S$ is irreducible then the second term in (3) vanishes, because $\mathfrak{u}_S^-$ is commutative. Thus in the complex $\mathbb{C} \otimes_{U(\mathfrak{u}_S^-)} D_*$ all differentials vanish and therefore

$$\dim(\mathrm{Tor}_j^{U(\mathfrak{u}_S^-)}(\mathbb{C}, \mathbb{C})) = \dim(\Lambda^j(\mathfrak{g}/\mathfrak{p}_S)).$$

Similarly the sequence (4) yields (cp. [14, Cor. 3.11])

$$\dim(\mathrm{Tor}_j^{U(\mathfrak{u}_S^-)}(\mathbb{C}, \mathbb{C})) = \sum_{\substack{w \in W^S, \\ l(w)=j}} \dim(M(w.0)).$$

Thus one obtains

$$\dim(\Lambda^j(\mathfrak{g}/\mathfrak{p}_S)) = \sum_{\substack{w\in W^S,\\ l(w)=j}} \dim(M(w.0)). \tag{5}$$

As $\Lambda^j(\mathfrak{g}/\mathfrak{p}_S)$ and $U(\mathfrak{g})$ are graded by the root lattice and the differentials d_j of the complex D_* respect this grading one can define a $\mathbb{Z}$-grading of the complex D_* by

$$\deg(u\otimes v) = (\omega_s, \mathrm{wt}(u)+\mathrm{wt}(v)) = (\omega_s, \mathrm{wt}(u)) - j$$

where $u \in U(\mathfrak{g})$ and $v \in \Lambda^j(\mathfrak{g}/\mathfrak{p}_S)$ are homogeneous elements. Similarly the complex C_* is $\mathbb{Z}$-graded by the same formula where now $v \in M(w.0)$ for some $w \in W^S$ with $l(w) = j$. Assume now that the complexes D_* and C_* are isomorphic as $\mathbb{Z}$-graded complexes of $U(\mathfrak{g})$-modules up to complex degree k. This holds for $k = 0$. Set $Z_k := \ker d_k \subset C_k = D_k$. Then $d_{k+1}(D_{k+1}) = Z_k = \delta_{k+1}(C_{k+1})$ by exactness of the sequences. Moreover, from (3) one obtains that d_{k+1} is injective when restricted to $1\otimes\Lambda^{k+1}(\mathfrak{g}/\mathfrak{p}_S)$. Note that

$$1\otimes\Lambda^{k+1}(\mathfrak{g}/\mathfrak{p}_S) = \{x\in D_{k+1} \mid \deg(x) = -(k+1)\}$$

and hence

$$d_{k+1}(1\otimes\Lambda^{k+1}(\mathfrak{g}/\mathfrak{p}_S)) = Z_k^{-(k+1)}$$

for $Z_k^{-(k+1)} := \{x \in Z_k \mid \deg(x) = -(k+1)\}$. As Z_k does not contain any element x such that $\deg(x) > -(k+1)$ one has $\delta_{k+1}(U(\mathfrak{g})\otimes_{U(\mathfrak{p}_S)} M(w.0)) = 0$ if $(\omega_s, \mathrm{wt}(w.0)) > -(k+1)$. By the injectivity of d_{k+1} and (5) one has

$$\dim(Z_k^{-(k+1)}) = \sum_{\substack{w\in W^S,\\ l(w)=k+1}} \dim(M(w.0)).$$

As δ_{k+1} maps onto Z_k this implies

$$\delta_{k+1}\left(1\otimes \bigoplus_{\substack{w\in W^S,\\ l(w)=k+1}} M(w.0)\right) = Z_k^{-(k+1)}.$$

Hence one obtains in view of (5) that the composition

$$\phi_{k+1} : 1\otimes \bigoplus_{\substack{w\in W^S,\\ l(w)=k+1}} M(w.0) \stackrel{\delta_{k+1}}{\longrightarrow} Z_k^{-(k+1)} \stackrel{(d_{k+1})^{-1}}{\longrightarrow} 1\otimes\Lambda^{k+1}(\mathfrak{g}/\mathfrak{p}_S)$$

is an isomorphism of $U(\mathfrak{p}_S)$-modules. By construction

$$\mathrm{Id}\otimes\phi_{k+1} : U(\mathfrak{g})\otimes_{U(\mathfrak{p}_S)} \bigoplus_{\substack{w\in W^S,\\ l(w)=k+1}} M(w.0) \to U(\mathfrak{g})\otimes_{U(\mathfrak{p}_S)} \Lambda^{k+1}(\mathfrak{g}/\mathfrak{p}_S)$$

extends the isomorphism of complexes to degree $k+1$. ■

As an application one obtains the following corollary.

Corollary 2.3. *Let $\mathfrak{g}/\mathfrak{p}_S$ be irreducible and let $w_1, w_2 \in W^S$ be elements of equal length $l(w_1) = l(w_2)$.*

(1) *One has $w_1.0 - w_2.0 \in Q_S$.*
(2) *If $w_1 \neq w_2$ then $w_1.0 - w_2.0 \notin Q_S^+$.*

Moreover, if $w, w' \in W^S$ and $l(w) = l(w')+1$ then $\omega_s(w.0 - w'.0) = 1$.

Proof. (1) By the above proposition $w_1.0$ and $w_2.0$ occur as weights of $\Lambda^{l(w_1)}(\mathfrak{g}/\mathfrak{p}_S)$. As $\mathfrak{g}/\mathfrak{p}_S$ is irreducible the weights of $\mathfrak{g}/\mathfrak{p}_S$ differ by elements in Q_S. Then so do the weights of $\Lambda^{l(w_1)}(\mathfrak{g}/\mathfrak{p}_S)$.

(2) Assume $w_1.0 - w_2.0 \in Q_S^+$, or equivalently $w_1\rho - w_2\rho \in Q_S^+$. Multiplication by w_2^{-1} and the definition of W^S yield

$$w_2^{-1} w_1 \rho - \rho \in Q^+.$$

Since ρ is dominant and W acts faithfully on ρ one obtains a contradiction unless $w_1 = w_2$.

The last statement follows from the fact that the map ϕ_k from the proof of the proposition is an isomorphism of $U(\mathfrak{p}_S)$-modules. ■

Remark 2.4. (1) In the above corollary the condition of irreducibility of $\mathfrak{g}/\mathfrak{p}_S$ can't be dropped. Indeed, for $S = \emptyset$ one has $Q_S = \{0\}$ but $w_1.0 \neq w_2.0$ for $w_1 \neq w_2$.

(2) Also one can't replace 0 by a more general weight $\mu \in P^+$. Consider for example $\mathfrak{g} = \mathfrak{sl}_4$, $S = \{\alpha_1, \alpha_3\}$, and $\mu = \omega_3$. Then $s_2 s_3.\omega_3 = \omega_3 - 2\alpha_3 - 3\alpha_2$ and $s_2 s_1.\omega_3 = \omega_3 - \alpha_1 - 2\alpha_2$ and hence $s_2 s_3.\omega_3 - s_2 s_1.\omega_3 = \alpha_1 - \alpha_2 - 2\alpha_3 \notin Q_S$. On the other hand $s_2 s_3$ and $s_2 s_1$ are elements of W^S of equal length.

3. Quantum groups

3.1. *Definition of $U_q(\mathfrak{g})$ and $\mathbb{C}_q[G]$*

We keep the notation of the previous section. Let $q \in \mathbb{C} \setminus \{0\}$ be not a root of unity. The q-deformed universal enveloping algebra $U_q(\mathfrak{g})$ associated to $\mathfrak{g}$ is considered here as the complex algebra generated by elements $K_i, K_i^{-1}, E_i, F_i, i = 1, \ldots, r$, and relations as given for instance in [12, 6.1.2]. In particular one has

$$\begin{aligned} K_i E_j &= q^{(\alpha_i,\alpha_j)} E_j K_i, \qquad K_i F_j = q^{-(\alpha_i,\alpha_j)} F_j K_i, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q^{(\alpha_i,\alpha_i)/2} - q^{-(\alpha_i,\alpha_i)/2}}. \end{aligned} \tag{6}$$

The algebra $U_q(\mathfrak{g})$ has a Hopf algebra structure with coproduct given by

$$\Delta K_i = K_i \otimes K_i, \qquad \Delta E_i = E_i \otimes K_i + 1 \otimes E_i, \qquad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i.$$

These formulae for the coproduct imply in particular that the antipode κ of $U_q(\mathfrak{g})$ is given by

$$\kappa(K_i) = K_i^{-1}, \qquad \kappa(E_i) = -E_i K_i^{-1}, \qquad \kappa(F_i) = -K_i F_i.$$

The counit will be denoted by ε. We will make frequent use of Sweedler notation in the form $\Delta u = u_{(1)} \otimes u_{(2)}$ for $u \in U_q(\mathfrak{g})$. Moreover, for any $u, x \in U_q(\mathfrak{g})$ we will write $(\operatorname{ad} u)x = u_{(1)} x \kappa(u_{(2)})$ to denote the left adjoint action.

There exists a uniquely determined algebra isomorphism coalgebra antiisomorphism η of $U_q(\mathfrak{g})$ such that

$$\eta(E_i) = F_i, \qquad \eta(F_i) = E_i, \quad \eta(K_i^{\pm 1}) = K_i^{\mp 1}.$$

Let $U_q(\mathfrak{n}^+), U_q(\mathfrak{n}^-) \subset U_q(\mathfrak{g})$ denote the subalgebras generated by $\{E_i \mid 1 \le i \le r\}$ and $\{F_i \mid 1 \le i \le r\}$, respectively. Let $U^0 \subset U_q(\mathfrak{g})$ be the subalgebra generated by $\{K_i, K_i^{-1} \mid 1 \le i \le r\}$. Moreover, let $G_+ \subset U_q(\mathfrak{g})$ denote the subalgebra generated by $\{E_i K_i^{-1} \mid 1 \le i \le r\}$.

For $\mu \in P^+$ let $V(\mu)$ denote the uniquely determined finite dimensional irreducible left $U_q(\mathfrak{g})$-module with highest weight μ. More explicitly, there exists a highest weight vector $v_\mu \in V(\mu) \setminus \{0\}$ satisfying

$$E_i v_\mu = 0, \qquad K_i v_\mu = q^{(\mu,\alpha_i)} v_\mu \quad \text{for all } i = 1, \ldots, r. \tag{7}$$

In general a vector $v \in V(\mu)$ is called a weight vector of weight $\mathrm{wt}(v) \in P$ if $K_i v = q^{(\mathrm{wt}(v),\alpha_i)} v$ for all $i = 1, \ldots, r$.

The dual V^* of a finite dimensional $U_q(\mathfrak{g})$-module V is defined as the dual vector space with the $U_q(\mathfrak{g})$-action given by

$$(uf)(v) = f(\kappa(u)v) \quad \forall v \in V, f \in V^*, u \in U_q(\mathfrak{g}).$$

For any left $U_q(\mathfrak{g})$-module V define a new $U_q(\mathfrak{g})$-module V_η to be the same vector space with the left $U_q(\mathfrak{g})$-module structure $\bullet_\eta$ given by

$$u\bullet_\eta v := \eta(u)v \quad \text{for all } u \in U_q(\mathfrak{g}), v \in V. \tag{8}$$

Note that $V(\mu)_\eta \cong V(\mu)^*$.

As usual the q-deformed coordinate ring $\mathbb{C}_q[G]$ is defined to be the subspace of the linear dual $U_q(\mathfrak{g})^*$ spanned by the matrix coefficients of the finite dimensional irreducible representations $V(\mu)$, $\mu \in P^+$. For $v \in V(\mu)$, $f \in V(\mu)^*$ the matrix coefficient $c^\mu_{f,v} \in U_q(\mathfrak{g})^*$ is defined by

$$c^\mu_{f,v}(X) = f(Xv).$$

The linear span of matrix coefficients of $V(\mu)$

$$C^{V(\mu)} = \mathrm{Lin}_\mathbb{C}\{c^\mu_{f,v} \mid v \in V(\mu), f \in V(\mu)^*\} \tag{9}$$

obtains a $U_q(\mathfrak{g})$-bimodule structure by

$$(Yc^\mu_{f,v}Z)(X) = f(ZXYv) = c^\mu_{fZ,Yv}(X). \tag{10}$$

Here $V(\mu)^*$ is considered as a right $U_q(\mathfrak{g})$-module. Note that by construction

$$\mathbb{C}_q[G] \cong \bigoplus_{\mu\in P^+} C^{V(\mu)} \tag{11}$$

is a Hopf algebra and the pairing

$$\mathbb{C}_q[G] \otimes U_q(\mathfrak{g}) \to \mathbb{C} \tag{12}$$

is nondegenerate.

3.2. Nilpotent and parabolic subalgebras

For $S \subset \pi$ let $U_q(\mathfrak{l}_S) \subset U_q(\mathfrak{g})$ denote the Hopf subalgebra generated by E_i, F_i, K_j, K_j^{-1} for all $\alpha_i \in S$ and all j. Moreover, let $V_- \subset U_q(\mathfrak{g})$ denote the subalgebra generated by the elements of the set

$$\{(\mathrm{ad}\, k)F_i \mid k \in U_q(\mathfrak{l}_S), \alpha_i \notin S\}.$$

Analogously, let $V_+ \subset U_q(\mathfrak{g})$ denote the subalgebra generated by the elements of the set

$$\{(\mathrm{ad}\, k)(E_iK_i^{-1}) \mid k \in U_q(\mathfrak{l}_S), \alpha_i \notin S\}.$$

As $(\mathrm{ad}\, E_i)F_j = 0 = (\mathrm{ad}\, F_i)(E_jK_j^{-1})$ for all $i \neq j$ one has $V_- \subset U_q(\mathfrak{n}^-)$ and $V_+ \subset G_+$. By [11, Prop. 4.2] multiplication gives isomorphisms

$$V_- \otimes U_q(\mathfrak{l}_S^-) \to U_q(\mathfrak{n}^-), \qquad V_+ \otimes U_q(\mathfrak{l}_S^+) \to G_+$$

where $U_q(\mathfrak{l}_S^-) := U_q(\mathfrak{n}^-) \cap U_q(\mathfrak{l}_S)$ and $U_q(\mathfrak{l}_S^+) := G_+ \cap U_q(\mathfrak{l}_S)$. Thus the triangular decomposition $U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^-) \otimes G_+ \otimes U^0$ yields

$$\begin{aligned} U_q(\mathfrak{g}) &\cong V_- \otimes U_q(\mathfrak{l}_S^-) \otimes V_+ \otimes U_q(\mathfrak{l}_S^+) \otimes U^0 \\ &\cong V_- \otimes V_+ \otimes U_q(\mathfrak{l}_S^-) \otimes U_q(\mathfrak{l}_S^+) \otimes U^0 \\ &\cong V_- \otimes V_+ \otimes U_q(\mathfrak{l}_S). \end{aligned} \tag{13}$$

Here in the second line the isomorphism

$$U_q(\mathfrak{l}_S^-) \otimes V_+ \to V_+ \otimes U_q(\mathfrak{l}_S^-), \qquad k \otimes v \mapsto (\mathrm{ad}\; k_{(1)})v \otimes k_{(2)}$$

is used, and the last line uses the triangular decomposition of $U_q(\mathfrak{l}_S)$. In a similar manner one obtains

$$U_q(\mathfrak{g}) \cong V_+ \otimes V_- \otimes U_q(\mathfrak{l}_S). \tag{14}$$

The parabolic subalgebra $U_q(\mathfrak{p}_S) \subset U_q(\mathfrak{g})$ is defined by

$$U_q(\mathfrak{p}_S) := \langle E_i, K_i, F_j \mid \alpha_i \in \pi, \alpha_j \in S \rangle.$$

Note that $U_q(\mathfrak{p}_S)$ coincides with the subalgebra generated by $U_q(\mathfrak{l}_S)$ and V_+. Thus by (13) multiplication yields isomorphisms

$$V_+ \otimes U_q(\mathfrak{l}_S) \cong U_q(\mathfrak{p}_S), \tag{15}$$
$$V_- \otimes U_q(\mathfrak{p}_S) \cong U_q(\mathfrak{g}). \tag{16}$$

4. Quantum generalized Verma modules

4.1. Notation

For $\lambda \in P_S^+$ as in the classical case $q = 1$ let $M(\lambda)$ denote the finite dimensional, irreducible $U_q(\mathfrak{l}_S)$-module of highest weight λ. Note that $M(\lambda)$ can be turned into an $U_q(\mathfrak{p}_S)$-module by setting $E_i v = 0$ for all generators E_i, $\alpha_i \notin S$, and $v \in M(\lambda)$.

Definition 4.1. For $\lambda \in P_S^+$, define the quantum generalized Verma module $V^{M(\lambda)}$ by

$$V^{M(\lambda)} := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}_S)} M(\lambda).$$

If $S = \emptyset$ and $\lambda \in P$ we will write $V^\lambda := V^{M(\lambda)}$.

Note that by (16) one has isomorphisms of U^0-modules $V^\lambda \cong U_q(\mathfrak{n}^-) \otimes \mathbb{C}^\lambda$ and $V^{M(\lambda)} \cong V_- \otimes M(\lambda)$ where $\mathbb{C}^\lambda$ denotes the one-dimensional U^0-module of weight λ.

Note moreover that $V_\eta^{M(\lambda)} \cong V^{M(\lambda)^*}$. Indeed, let $\xi_{-\lambda} \in M(\lambda)^*$ denote the up to scalar multiplication uniquely determined element of weight $-\lambda$. Then $1 \otimes \xi_{-\lambda} \in V^{M(\lambda)^*}$ is a cyclic vector and a set of relations determining $V^{M(\lambda)^*}$ is given by

$$K_j^{\pm 1}(1 \otimes \xi_{-\lambda}) = q^{\mp(\lambda,\alpha_j)} 1 \otimes \xi_{-\lambda}, \qquad E_i^{(\lambda,\alpha_i^\vee)+1}(1 \otimes \xi_{-\lambda}) = 0$$

for all $\alpha_i \in S$ and for all j. The same relations hold for the cyclic vector $1 \otimes v_\lambda \in V_\eta^{M(\lambda)}$.

Remark. The notation used here slightly differs from the original notation in [14]. Recall that ρ denotes half the sum of the positive roots and define

$$\rho_S := \frac{1}{2} \sum_{\alpha \in R_S^+} \alpha.$$

Note that for all $\lambda \in P$ one has

$$\lambda \in P_S^+ \Leftrightarrow \lambda - \rho + \rho_S \in P_S^+.$$

Lepowsky considered modules obtained by twisted induction in the classical case $q = 1$ and defined $V^{M(\lambda)} := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_S)} M(\lambda - \rho + \rho_S)$. Translation between the two settings is straightforward.

Let $\lambda \in P_S^+$ and let $v_\lambda \in M(\lambda)$ denote a vector of weight λ. For any $U_q(\mathfrak{g})$-module homomorphism $g : V^{M(\lambda)} \to V^{M(\mu)}$ there exists an element $F \in U_q(\mathfrak{n}^-)$ such that $g(1 \otimes v_\lambda) = F \otimes v_\mu$. Note that F is uniquely determined up to addition of an element in the annihilator of $1 \otimes v_\mu \in V^{M(\mu)}$. We will say that the homomorphism g *is determined by* F.

4.2. The Bernstein–Gelfand–Gelfand resolution

We now briefly recall the quantum analogue of the Bernstein–Gelfand–Gelfand resolution. This construction has been in detail considered in [4] for q not a root of unity.

Fix a dominant integral weight $\mu \in P^+$. For all $j = 0, \ldots, \dim(\mathfrak{g}/\mathfrak{p}_S)$ define

$$C_j^S := \bigoplus_{w \in W^S,\, l(w)=j} V^{M(w.\mu)}.$$

Note that $V^{M(w.\mu)}$ is a highest weight module with highest weight $w.\mu$. Therefore $V^{M(w.\mu)}$ is a natural quotient of $V^{w.\mu}$.

As in [14, Section 4] one constructs $U_q(\mathfrak{g})$-module maps $\varphi_j^S : C_j^S \to C_{j-1}^S$ for all $j = 1, \ldots, \dim(\mathfrak{g}/\mathfrak{p}_S)$. More explicitly, for all $w \in W$, fix an embedding $V^{w.\mu} \subset V^\mu$. Then for all $w, w' \in W$ with $w \le w'$ one has a fixed embedding $f_{w,w'} : V^{w.\mu} \to V^{w'.\mu}$.

A quadruple (w_1, w_2, w_3, w_4) of elements of W is called a square if $w_2 \neq w_3$ and

$$w_1 \to w_2 \to w_4, \quad w_1 \to w_3 \to w_4.$$

By [2, Lemma 10.4] to each arrow $w_1 \to w_2$ ($w_1, w_2 \in W$) one can assign a number $s(w_1, w_2) = \pm 1$ such that for every square, the product of the numbers assigned to the four arrows occurring in it is -1. Let $w, w' \in W^S$ such that $l(w) = l(w') + 1$. If $w \to w'$ then let $h_{w,w'} : V^{M(w.\mu)} \to V^{M(w'.\mu)}$ denote the (standard) map induced by the map

$$s(w, w') f_{w,w'} : V^{w.\mu} \to V^{w'.\mu}.$$

Otherwise, define $h_{w,w'} = 0$. The differential φ_j^S is now defined as the sum of all $h_{w,w'}$ where $l(w) = j = l(w') + 1$.

Note moreover, that for $\mu \in P^+$ there exists a surjective map of $U_q(\mathfrak{g})$-modules

$$\varepsilon_\mu : C_0^S = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}_S)} M(\mu) \to V(\mu), \quad u \otimes v_{\mu,M} \mapsto u v_{\mu,V} \tag{17}$$

where $v_{\mu,M} \in M(\mu)$ and $v_{\mu,V} \in V(\mu)$ denote vectors of weight μ.

Theorem 4.2 (*[4, Section 3.4]*). *The sequence*

$$0 \longrightarrow C^S_{\dim \mathfrak{g}/\mathfrak{p}_S} \xrightarrow{\varphi^S_{\dim(\mathfrak{g}/\mathfrak{p}_S)}} \cdots \xrightarrow{\varphi_1^S} C_0^S \xrightarrow{\varepsilon_\mu} V(\mu) \longrightarrow 0 \tag{18}$$

is exact and $\varphi_j^S(V^{M(w.\mu)}) \neq 0$ *for all* $j = 1, \ldots, \dim(\mathfrak{g}/\mathfrak{p}_S)$ *and all* $w \in W^S$ *with* $l(w) = j$.

Remark 4.3. In the quantum case the fact that for $w \to w'$ the standard map $h_{w,w'} : V^{M(w.\mu)} \to V^{M(w'.\mu)}$ is nonzero has not been explicitly stated in [4]. However, this property can be verified analogously to formula (1) in the proof of [13, Lemma 9.2.14]. The necessary fact that for $\mu, \lambda \in \mathfrak{h}^*$ the simple module $V(\mu)$ is a subquotient of V^λ if and only if $\mathrm{Hom}(V^\mu, V^\lambda) \neq 0$ follows as in [17] after translation of [19, Sections 1–6] to the quantum case.

By construction there exists $y^\mu_{w,w'} \in U_q(\mathfrak{n}^-)$ such that $f_{w,w'}(u \otimes v_{w.\mu}) = u y^\mu_{w,w'} \otimes v_{w'.\mu}$. Thus in terms of the elements $y^\mu_{w,w'} \in U_q(\mathfrak{n}^-)$ the map $h_{w,w'}$ is given by

$$h_{w,w'}(u \otimes v_{w.\mu}) = s(w, w') u y^\mu_{w,w'} \otimes v_{w'.\mu}.$$

In later considerations the main focus will be on the case $\mu = 0$. In this case define $y_{w,w'} := s(w, w') y^0_{w,w'} \in U_q(\mathfrak{n}^-)$.

5. $U_q(\mathfrak{g})$-modules induced by $U_q(\mathfrak{l}_S)$-modules

5.1. Notation

Definition 5.1. For $\lambda \in P_S^+$, define

$$W^{M(\lambda)} := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} M(\lambda).$$

If $S = \emptyset$ and $\lambda \in P$ we will write $W^\lambda := W^{M(\lambda)}$.

Note that multiplication yields isomorphisms $W^\lambda \cong U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{n}^+) \otimes \mathbb{C}^\lambda$ and $W^{M(\lambda)} \cong V_- \otimes V_+ \otimes M(\lambda)$ of U^0-modules. Note moreover that $W^{M(\lambda)}$ is in general not a highest weight module. In analogy to the observation after Definition 4.1 one obtains $W_\eta^{M(\lambda)} \cong W^{M(\lambda)^*}$.

5.2. *The functor* $\_ : \mathcal{V} \to \mathcal{W}$

By Definitions 4.1 and 5.1 there exists a natural surjective $U_q(\mathfrak{g})$-module homomorphism

$$\Phi_\lambda : W^{M(\lambda)} \to V^{M(\lambda)}.$$

Proposition 5.2. *For any $U_q(\mathfrak{g})$-module homomorphism $g : V^{M(\lambda)} \to V^{M(\mu)}$ there exists a uniquely determined $U_q(\mathfrak{g})$-module homomorphism $\underline{g} : W^{M(\lambda)} \to W^{M(\mu)}$ such that the diagram*

$$\begin{array}{ccc} W^{M(\lambda)} & \xrightarrow{\underline{g}} & W^{M(\mu)} \\ \downarrow{\scriptstyle \Phi_\lambda} & & \downarrow{\scriptstyle \Phi_\mu} \\ V^{M(\lambda)} & \xrightarrow{g} & V^{M(\mu)} \end{array} \qquad (19)$$

commutes.

Proof. Assume that g is determined by $F \in U_q(\mathfrak{n}^-)$ as in the end of Section 4.1. To obtain commutativity of the diagram (19) one has to define $\underline{g}(1 \otimes v_\lambda) = F \otimes v_\mu$. We have to check that $\underline{g}$ is well defined. To this end consider $0 = u \otimes v_\lambda \in W^{M(\lambda)}$, or equivalently $u \in U_q(\mathfrak{g})\mathrm{Ann}_{U_q(\mathfrak{l}_S)}(v_\lambda)$. We have to show that $uF \in U_q(\mathfrak{g})\mathrm{Ann}_{U_q(\mathfrak{l}_S)}(v_\mu)$.

Using the decomposition (14) and the fact that $V_-U_q(\mathfrak{l}_S)F \subset V_-U_q(\mathfrak{l}_S)$ one may assume that $u \in V_-U_q(\mathfrak{l}_S)$. The relation $g(\Phi_\lambda(u \otimes v_\lambda)) = 0$ implies $uF \in U_q(\mathfrak{g})\mathrm{Ann}_{U_q(\mathfrak{p}_S)}(v_\mu)$. Hence

$$\begin{aligned} uF \in \;& U_q(\mathfrak{g})\mathrm{Ann}_{U_q(\mathfrak{p}_S)}(v_\mu) \cap V_-U_q(\mathfrak{l}_S) \\ \overset{(16)}{=}\;& V_-(\mathrm{Ann}_{U_q(\mathfrak{p}_S)}(v_\mu) \cap U_q(\mathfrak{l}_S)) = V_-\mathrm{Ann}_{U_q(\mathfrak{l}_S)}(v_\mu). \quad \blacksquare \end{aligned}$$

Let $\mathcal{V}$ and $\mathcal{W}$ denote the full subcategory of the category of $U_q(\mathfrak{g})$-modules whose objects are finite direct sums of $U_q(\mathfrak{g})$-modules $V^{M(\lambda)}$ and $W^{M(\lambda)}$, where $\lambda \in P_S^+$, respectively. By Proposition 5.2 there exists a canonical functor $\_ : \mathcal{V} \to \mathcal{W}$ such that

$$\underline{\bigoplus_{i=1}^n V^{M(\lambda_i)}} = \bigoplus_{i=1}^n W^{M(\lambda_i)}.$$

Proposition 5.3. *The functor $\_ : \mathcal{V} \to \mathcal{W}$ is exact.*

Proof. Recall that $V^{M(\lambda)} \cong V_- \otimes M(\lambda)$ and $W^{M(\lambda)} \cong V_+ \otimes V_- \otimes M(\lambda)$. With respect to these decompositions one gets for any $V_1, V_2 \in \mathcal{V}$ and any $g : V_1 \to V_2$ the relation $\underline{g} = \mathrm{Id}_{V_+} \otimes g$. Hence $\_$ preserves exactness. $\blacksquare$

Let $U_q(\mathfrak{p}_S^{\mathrm{op}}) \subset U_q(\mathfrak{g})$ denote the subalgebra generated by the elements E_j, K_i, F_i for $\alpha_i \in \pi, \alpha_j \in S$. For any $\mu \in P^+$ define a map

$$\begin{aligned} \underline{\varepsilon}_\mu : W^{M(\mu)} = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} M(\mu) &\to U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}_S^{\mathrm{op}})} V(\mu) \\ u \otimes v_{\mu,M} &\mapsto u \otimes v_{\mu,V} \end{aligned}$$

where as in (17) the symbols $v_{\mu,M} \in M(\mu)$ and $v_{\mu,V} \in V(\mu)$ denote vectors of weight μ. Then by the same argument as in the proof of Proposition 5.3 the BGG resolution (18) induces an exact sequence

$$0 \longrightarrow \underline{C}^S_{\dim \mathfrak{g}/\mathfrak{p}_S} \xrightarrow{\underline{\varphi}^S_{\dim(\mathfrak{g}/\mathfrak{p}_S)}} \cdots \xrightarrow{\underline{\varphi}^S_1} \underline{C}^S_0 \xrightarrow{\underline{\varepsilon}_\mu} U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}_S^{\mathrm{op}})} V(\mu) \longrightarrow 0. \qquad (20)$$

5.3. Homomorphisms and estimates

Note that for $\mu, \nu \in P_S^+$ the left $U_q(\mathfrak{l}_S)$-module $M(\mu) \otimes M(\nu)^*$ is generated by one element $v_\mu \otimes \xi_{-\nu}$ where $v_\mu \in M(\mu)$ and $\xi_{-\nu} \in M(\nu)^*$ denote a highest and a lowest weight vector, respectively. A complete set of relations for $M(\mu) \otimes M(\nu)^*$ is given by

$$\begin{aligned} &E_i^{(\nu,\alpha_i^\vee)+1}(v_\mu \otimes \xi_{-\nu}) = 0,\\ &F_i^{(\mu,\alpha_i^\vee)+1}(v_\mu \otimes \xi_{-\nu}) = 0,\\ &(K_j - q^{(\mu-\nu,\alpha_j)})(v_\mu \otimes \xi_{-\nu}) = 0 \end{aligned} \tag{21}$$

where $\alpha_i \in S$ and $\alpha_j \in \pi$. This follows for instance from [10, Prop. 5.2] using the fact that the module generated by one element and the relations (21) is integrable and the generator is a cyclic weight vector. In Section 7.4 we will be interested in homomorphisms between $U_q(\mathfrak{g})$-modules induced by $U_q(\mathfrak{l}_S)$-modules $M(\mu) \otimes M(\nu)^*$. Here we derive well definedness and some properties of such maps.

For $w, w' \in W^S$, $w \to w'$, and $\mu \in P^+$ recall the definition of the element $y^\mu_{w,w'} \in U_q(\mathfrak{n}^-)$ from Section 4.2 and define $x^\mu_{w,w'} := \eta(y^\mu_{w,w'})$. Define $U_q(\mathfrak{l}_S)$-module homomorphisms

$$\begin{aligned} &\theta_2 : M(w.\mu) \to V^{M(w'.\mu)} \cong V_+V_- \otimes M(w'.\mu),\\ &uv_{w.\mu} \mapsto uy^\mu_{w,w'} \otimes v_{w'.\mu},\\ &\overline{\theta}_2 : M(w.\mu)^* \to V^{M(w'.\mu)^*} \cong V_-V_+ \otimes M(w'.\mu)^*,\\ &u\xi_{-w.\mu} \mapsto ux^\mu_{w,w'} \otimes \xi_{-w'.\mu}. \end{aligned}$$

Proposition 5.4. *Let $w, w' \in W^S$, $w \to w'$, $\mu \in P^+$, and $\nu \in P_S^+$. There are uniquely determined injective $U_q(\mathfrak{l}_S)$-module homomorphisms*

$$\begin{aligned} &\theta_1 : M(w.\mu) \otimes M(\nu)^* \to U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(w'.\mu) \otimes M(\nu)^*),\\ &\overline{\theta}_1 : M(\nu) \otimes M(w.\mu)^* \to U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(\nu) \otimes M(w'.\mu)^*) \end{aligned}$$

such that

$$\theta_1(v_{w.\mu} \otimes \xi_{-\nu}) = y^\mu_{w,w'} \otimes (v_{w'.\mu} \otimes \xi_{-\nu}), \tag{22}$$

$$\overline{\theta}_1(v_\nu \otimes \xi_{-w.\mu}) = x^\mu_{w,w'} \otimes (v_\nu \otimes \xi_{-w'.\mu}). \tag{23}$$

Moreover, in $V_+V_- \otimes M(w'.\mu) \otimes M(\nu)^*$ one has for all weight vectors $v \in M(w.\mu)$, $\xi \in M(\nu)^*$

$$\theta_1(v \otimes \xi) \in \theta_2(v) \otimes \xi + \sum_{\zeta < \mathrm{wt}(\xi)} V_- \otimes M(w'.\mu) \otimes M(\nu)^*_\zeta. \tag{24}$$

Similarly, in $V_-V_+ \otimes M(\nu) \otimes M(w'.\mu)^*$ one has for all weight vectors $v \in M(\nu)$, $\xi \in M(w.\mu)^*$

$$\overline{\theta}_1(v \otimes \xi) \in P_{23}(\overline{\theta}_2(\xi) \otimes v) + \sum_{\zeta > \mathrm{wt}(v)} V_+ \otimes M(\nu)_\zeta \otimes M(w'.\mu)^* \tag{25}$$

where P_{23} denotes the flip of the second and the third tensor factor.

Proof. The maps θ_1 and $\overline{\theta}_1$ are uniquely determined by formulae (22) and (23), respectively. It remains to verify that they are well defined and injective. Fix $\alpha_i \in S$ and let $U_i \subset U_q(\mathfrak{g})$ denote the subalgebra isomorphic to $U_q(\mathfrak{sl}_2)$ generated by E_i, F_i, and $K_i^{\pm 1}$. Note that

$$F_i^{(w.\mu,\alpha_i^\vee)+1} y^\mu_{w,w'} \otimes (v_{w'.\mu} \otimes \xi_{-\nu}) = 0 \tag{26}$$

for all $\alpha_i \in S$. Indeed, as the standard map $h_{w,w'}$ is well defined one obtains

$$F_i^{(w.\mu,\alpha_i^\vee)+1} y^\mu_{w,w'} \in (U_q(\mathfrak{g})\mathrm{Ann}_{U_q(\mathfrak{l}_S)} v_{w'.\mu}) \cap U_q(\mathfrak{n}^-).$$

Hence (26) follows from the fact that $\xi_{-\nu}$ is a lowest weight vector.

Note that $U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(w.\mu) \otimes M(\nu)^*)$ is an integrable $U_q(\mathfrak{l}_S)$-module. Hence the weight vector $y^\mu_{w,w'} \otimes (v_{w'.\mu} \otimes \xi_{-\nu})$ can be written as a sum of weight vectors of weight $w.\mu - \nu$ which generate pairwise nonisomorphic irreducible U_i-modules. By (26) among these irreducible U_i-modules there is one of lowest weight $w.\mu - \nu - (w.\mu, \alpha_i^\vee)\alpha_i$ and all other U_i-modules generated by $y^\mu_{w,w'} \otimes v_{w'.\mu} \otimes \xi_{-\nu}$ have larger lowest weight. The corresponding highest weight with respect to U_i is $\nu - w.\mu + (w.\mu, \alpha_i^\vee)\alpha_i = w.\mu - \nu + (\nu, \alpha_i^\vee)\alpha_i$ and hence

$$E_i^{(\nu,\alpha_i^\vee)+1} y^\mu_{w,w'} \otimes (v_{w'.\mu} \otimes \xi_{-\nu}) = 0.$$

In view of (21) this proves that θ_1 is well defined. The injectivity of θ_1 will follow from (24).

To prove (24) note that for any weight vectors $v \in M(w.\mu)_\lambda$ and $\xi \in M(\nu)^*_{\lambda'}$ there exist $E \in U_q(\mathfrak{l}_S^+)_{\lambda'+\nu}$ and $F \in U_q(\mathfrak{l}_S^-)_{w.\mu-\lambda}$ such that $v = F v_{w.\mu}$ and $\xi = E\xi_{-\nu}$. Moreover, in the $U_q(\mathfrak{l}_S)$-module $M(w.\mu) \otimes M(\nu)^*$ one has

$$v \otimes \xi \in FE(v_{w.\mu} \otimes \xi_{-\nu}) + \sum_{\zeta < \mathrm{wt}(\xi)} M(w.\mu) \otimes M(\nu)^*_\zeta.$$

Hence one obtains by induction on $\mathrm{wt}(\xi)$

$$\begin{aligned}\theta_1(v \otimes \xi) &\in FEy^\mu_{w,w'} \otimes (v_{w'.\mu} \otimes \xi_{-\nu}) + \sum_{\zeta < \mathrm{wt}(\xi)} V_- \otimes M(w.\mu) \otimes M(\nu)^*_\zeta \\ &= Fy^\mu_{w,w'} \otimes (v_{w'.\mu} \otimes \xi) + \sum_{\zeta < \mathrm{wt}(\xi)} V_- \otimes M(w.\mu) \otimes M(\nu)^*_\zeta \\ &= \theta_2(v) \otimes \xi + \sum_{\zeta < \mathrm{wt}(\xi)} V_- \otimes M(w.\mu) \otimes M(\nu)^*_\zeta.\end{aligned}$$

The well definedness and the injectivity of $\overline{\theta}_1$ follow from the corresponding properties of θ_1 and the relations

$$(M(w'.\mu) \otimes M(\nu)^*)_\eta \cong M(\nu) \otimes M(w'.\mu)^*,$$
$$\left(U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(w'.\mu) \otimes M(\nu)^*)\right)_\eta \cong U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(\nu) \otimes M(w'.\mu)^*).$$

Formula (25) is proved in the same manner as (24). ■

For any $\mu, \nu \in P^+_S$ we define

$$W(\mu,\nu) := U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(\mu) \otimes M(\nu)^*).$$

Using the isomorphism

$$W(\mu,\nu) \cong V_+V_- \otimes M(\mu) \otimes M(\nu)^* \cong V_-V_+ \otimes M(\mu) \otimes M(\nu)^*$$

we define two filtration on $W(\mu,\nu)$ as follows

$$\mathcal{F}_1^k W(\mu,\nu) = \mathrm{Lin}_\mathbb{C}\{u \otimes v \otimes \xi \mid u \in V_-V_+, v \in M(\mu)_\lambda, \xi \in M(\nu)^*, \mathrm{ht}(\mu - \lambda) \le k\}, \quad (27)$$

$$\mathcal{F}_2^k W(\mu,\nu) = \mathrm{Lin}_\mathbb{C}\{u \otimes v \otimes \xi \mid u \in V_+V_-, v \in M(\mu), \xi \in M(\nu)^*_\lambda, \mathrm{ht}(\lambda + \nu) \le k\}. \quad (28)$$

Corollary 5.5. *Assume that $w, w' \in W^S$, $w \to w'$, $\mu \in P^+$, $\nu \in P^+_S$. The following relation holds in $W(w'.\mu,\nu)$*

$$U_q(\mathfrak{g})y^\mu_{w,w'} \otimes (v_{w'.\mu} \otimes \xi_{-\nu}) \cap \mathcal{F}_2^k W(w'.\mu,\nu) \subseteq \sum_{\mathrm{ht}(\beta)\le k} V_+ U_q(\mathfrak{n}^-) U_q(\mathfrak{l}_S^+)_\beta y^\mu_{w,w'} \otimes (v_{w'.\mu} \otimes \xi_{-\nu}).$$

Similarly one has in $W(\nu, w'.\mu)$ the relation

$$U_q(\mathfrak{g})x^\mu_{w,w'} \otimes (v_\nu \otimes \xi_{-w'.\mu}) \cap \mathcal{F}_1^k W(w'.\mu,\nu) \subseteq \sum_{\mathrm{ht}(\beta)\le k} V_- U_q(\mathfrak{n}^+) U_q(\mathfrak{l}_S^-)_{-\beta} x^\mu_{w,w'} \otimes (v_\nu \otimes \xi_{-w'.\mu}).$$

Proof. Proposition 5.4 implies the following equalities.

$$\begin{aligned}
V_+V_-\theta_1(M(w.\mu)\otimes M(\nu)^*)\cap\ &\mathcal{F}_2^k W(w'.\mu,\nu)\\
&\overset{(24)}{=} V_+V_-\theta_1\left(\sum_{\mathrm{ht}(\alpha+\nu)\le k} M(w.\mu)\otimes M(\nu)^*_\alpha\right)\\
&= V_+V_-\theta_1\left(\sum_{\mathrm{ht}(\beta)\le k} U_q(\mathfrak{l}_S^-)U_q(\mathfrak{l}_S^+)_\beta(v_{w.\mu}\otimes\xi_{-\nu})\right)\\
&= \sum_{\mathrm{ht}(\beta)\le k} V_+U_q(\mathfrak{n}^-)U_q(\mathfrak{l}_S^+)_\beta y^\mu_{w,w'}\otimes(v_{w.\mu}\otimes\xi_{-\nu}).
\end{aligned}$$

The second relation is verified analogously. ■

Corollary 5.6. *Assume that $w, w'\in W^S$, $w\to w'$, $\nu\in P_S^+$, and $x\in U_q(\mathfrak{n}^+)$. Then in $W(w'.0,\nu)$ the relation*

$$[y_{w,w'},x]\otimes(v_{w'.0}\otimes\xi_{-\nu})\notin\sum_{\substack{w''\in W^S,\\ w''\to w'}} U_q(\mathfrak{g})y_{w'',w'}\otimes(v_{w'.0}\otimes\xi_{-\nu})\setminus\{0\} \tag{29}$$

holds. Similarly, for $y\in U_q(\mathfrak{n}^-)$ the relation

$$[x_{w,w'},y]\otimes(v_\nu\otimes\xi_{-w'.0})\notin\sum_{\substack{w''\in W^S,\\ w''\to w'}} U_q(\mathfrak{g})x_{w'',w'}\otimes(v_\nu\otimes\xi_{-w'.0})\setminus\{0\} \tag{30}$$

holds in $W(\nu,w'.0)$

Proof. Recall that $y_{w,w'}\in U_q(\mathfrak{n}^-)_{w.0-w'.0}$. Hence with respect to the decomposition

$$U_q(\mathfrak{g})\cong V_+\otimes U_q(\mathfrak{n}^-)\otimes U_q(\mathfrak{l}_S^+)\otimes U^0 \tag{31}$$

one obtains using (6)

$$[y_{w,w'},x]\in\sum_{\beta>w.0-w'.0} V_+\otimes U_q(\mathfrak{n}^-)_\beta\otimes U_q(\mathfrak{l}_S^+)\otimes U^0.$$

As U^0 acts diagonally and $v_{w'.0}$ is a highest weight vector for the action of $U_q(\mathfrak{l}_S)$ this implies

$$[y_{w,w'},x]\otimes(v_{w'.0}\otimes\xi_{-\nu})\in\sum_{\alpha+\beta>w.0} V_+V_{-,\alpha}\otimes M(w'.0)_\beta\otimes M(\nu)^*. \tag{32}$$

On the other hand, for any $k\in\mathbb{N}_0$, Corollary 5.5 implies

$$\begin{aligned}
&\sum_{\substack{w''\in W^S,\\ w''\to w'}} U_q(\mathfrak{g})y_{w'',w'}\otimes(v_{w'.0}\otimes\xi_{-\nu})\cap\mathcal{F}_2^k W(w'.0,\nu)\\
&\quad\subseteq\sum_{\substack{w''\in W^S,\\ w''\to w'}}\sum_{\mathrm{ht}(\gamma)\le k} V_+U_q(\mathfrak{n}^-)U_q(\mathfrak{l}_S^+)_\gamma y_{w'',w'}\otimes(v_{w'.0}\otimes\xi_{-\nu})\\
&\quad\subseteq\sum_{\substack{w''\in W^S,\\ w''\to w'}}\sum_{\mathrm{ht}(\gamma+\nu)\le k} V_+U_q(\mathfrak{n}^-)y_{w'',w'}\otimes(v_{w'.0}\otimes M(\nu)^*_\gamma)+\mathcal{F}_2^{k-1}W(w'.0,\nu)\\
&\quad\subseteq\sum_{\substack{w''\in W^S,\\ w''\to w'}}\sum_{\alpha+\beta\le w''.0} V_+V_{-,\alpha}\otimes M(w'.0)_\beta\otimes M(\nu)^*+\mathcal{F}_2^{k-1}W(w'.0,\nu).
\end{aligned} \tag{33}$$

Choose now $k\in\mathbb{N}$ such that $[y_{w,w'},x]\otimes(v_{w'.0}\otimes\xi_{-\nu})\in\mathcal{F}_2^k W(w'.0,\nu)\setminus\mathcal{F}_2^{k-1}W(w'.0,\nu)$ and assume that (29) does not hold. Then (32) and (33) imply that there exists $w''\in W^S$, $w''\to w'$ such that $w''.0>w.0$. This is a contradiction to Corollary 2.3(1) and (2). Hence (29) holds. Relation (30) is verified analogously. ■

6. Categorical equivalence

From now on we will write $\mathcal{A} = \mathbb{C}_q[G]$ and

$$\mathcal{B} = \{b \in \mathcal{A} \mid b_{(1)} b_{(2)}(k) = \varepsilon(k) b \text{ for all } k \in U_q(\mathfrak{l}_S)\}. \tag{34}$$

6.1. Takeuchi's categorical equivalence

In this subsection Takeuchi's categorical equivalence [23] is recalled in the present special setting. Note that $\mathcal{B} \subset \mathcal{A}$ is a left coideal subalgebra of the Hopf algebra $\mathcal{A}$. Thus $\overleftarrow{\mathcal{A}} := \mathcal{A}/\mathcal{B}^+\mathcal{A}$ where $\mathcal{B}^+ = \{b \in \mathcal{B} \mid \varepsilon(b) = 0\}$ is a right $\mathcal{A}$-module coalgebra. Moreover, by [16, Thm. 2.2(2)] $\mathcal{A}$ is a faithfully flat right $\mathcal{B}$-module. It was shown in the proof of [16, Thm. 2.2(1), (2)] that $\overleftarrow{\mathcal{A}}$ is equal to the image of $\mathcal{A}$ under the restriction map $U_q(\mathfrak{g})^\circ \to U_q(\mathfrak{l}_S)^\circ$ of dual Hopf algebras. Therefore the pairing

$$\langle \cdot, \cdot \rangle : U_q(\mathfrak{l}_S) \times \overleftarrow{\mathcal{A}} \to \mathbb{C} \tag{35}$$

is nondegenerate. Let ${}^{\overleftarrow{\mathcal{A}}}\mathcal{M}$ denote the category of finite dimensional left $\overleftarrow{\mathcal{A}}$-comodules and let $\mathcal{M}_{U_q(\mathfrak{l}_S)}$ denote the category of right $U_q(\mathfrak{l}_S)$-modules which are isomorphic to a finite direct sum of modules of the form $M(\lambda)^*$, $\lambda \in P^S_+$. The pairing (35) induces a functor

$$\Xi : {}^{\overleftarrow{\mathcal{A}}}\mathcal{M} \to \mathcal{M}_{U_q(\mathfrak{l}_S)} \tag{36}$$

where for $V \in {}^{\overleftarrow{\mathcal{A}}}\mathcal{M}$ the right $U_q(\mathfrak{l}_S)$-module structure on $\Xi(V) := V$ is given by $v \triangleleft k = \langle k, v_{(-1)} \rangle v_{(0)}$ for all $k \in U_q(\mathfrak{l}_S)$, $v \in V$.

Proposition 6.1. *The functor Ξ is an equivalence of categories.*

Proof. By the nondegeneracy of the pairing (35) two objects $V, W \in {}^{\overleftarrow{\mathcal{A}}}\mathcal{M}$ are isomorphic if and only if the $U_q(\mathfrak{l}_S)$-modules $\Xi(V)$ and $\Xi(W)$ are isomorphic. It remains to show that all objects of $\mathcal{M}_{U_q(\mathfrak{l}_S)}$ lie in the image of Ξ. To this end, consider the right $U_q(\mathfrak{l}_S)$-module $M(\lambda)^*$, $\lambda \in P^S_+$. By Lemma 2.1 one can find $\mu \in P_+$ and an embedding of $U_q(\mathfrak{l}_S)$-modules $M(\lambda) \hookrightarrow V(\mu)$. Then $V(\mu)^*$ is a right $U_q(\mathfrak{g})$-module, or equivalently by definition of $\mathcal{A}$, a left $\mathcal{A}$-comodule. Projection onto $\overleftarrow{\mathcal{A}}$ endows $V(\mu)^*$ with a left $\overleftarrow{\mathcal{A}}$-comodule structure. As $V(\mu)$ decomposes into a direct sum of irreducible $U_q(\mathfrak{l}_S)$-modules the $U_q(\mathfrak{l}_S)$-module $M(\lambda)^*$ can be viewed as a direct summand of the $U_q(\mathfrak{l}_S)$-module $V(\mu)^*$. As the pairing (35) is nondegenerate the $U_q(\mathfrak{l}_S)$-direct summand $M(\lambda)^* \subset V(\mu)^*$ is an $\overleftarrow{\mathcal{A}}$-subcomodule. By construction, application of Ξ to this $\overleftarrow{\mathcal{A}}$-subcomodule yields the right $U_q(\mathfrak{l}_S)$-module $M(\lambda)^*$. ■

Recall that for any coalgebra C the cotensor product of a right C-comodule P and a left C-comodule Q is defined by

$$P \square_C Q := \left\{ \sum_i p_i \otimes q_i \in P \otimes Q \,\middle|\, \sum_i p_{i(0)} \otimes p_{i(1)} \otimes q_i = \sum_i p_i \otimes q_{i(-1)} \otimes q_{i(0)} \right\}.$$

Let ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$ denote the category of left $\mathcal{A}$-covariant left $\mathcal{B}$-modules. There exist functors

$$\Phi : {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M} \to {}^{\overleftarrow{\mathcal{A}}}\mathcal{M}, \quad \Phi(\Gamma) = \Gamma/\mathcal{B}^+\Gamma, \tag{37}$$

$$\Psi : {}^{\overleftarrow{\mathcal{A}}}\mathcal{M} \to {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}, \quad \Psi(V) = \mathcal{A} \square_{\overleftarrow{\mathcal{A}}} V. \tag{38}$$

Here for any $\Gamma \in {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$ the left $\overleftarrow{\mathcal{A}}$-comodule structure on $\Gamma/\mathcal{B}^+\Gamma$ is induced by the left $\mathcal{A}$-comodule structure of Γ. Moreover, the left $\mathcal{B}$-module and the left $\mathcal{A}$-comodule structures of $\mathcal{A} \square_{\overleftarrow{\mathcal{A}}} V$ are defined on the first tensor factor.

Theorem 6.2 ([23, Theorem 1]). *With the notions as above Φ and Ψ are mutually inverse equivalences of categories.*

By the above theorem and Proposition 6.1 in order to show that two $\mathcal{A}$-covariant $\mathcal{B}$-modules coincide, it suffices to show that the corresponding $U_q(\mathfrak{l}_S)$-modules coincide. This method will be applied to show that the differential graded algebra which will be constructed in Section 7 coincides with the q-deformed de Rham complex constructed in [7].

A slight refinement of Theorem 6.2 also takes into account possible right $\mathcal{B}$-modules structures. Let ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}}$ and ${}^{\overleftarrow{\mathcal{A}}}\mathcal{M}_{\mathcal{B}}$ denote the categories of left $\mathcal{A}$-covariant $\mathcal{B}$-bimodules and of left $\overleftarrow{\mathcal{A}}$-covariant right $\mathcal{B}$-modules, respectively. The functors Φ and Ψ restrict to functors $\Phi_{\mathcal{B}} : {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}} \to {}^{\overleftarrow{\mathcal{A}}}\mathcal{M}_{\mathcal{B}}$ and $\Psi_{\mathcal{B}} : {}^{\overleftarrow{\mathcal{A}}}\mathcal{M}_{\mathcal{B}} \to {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}_{\mathcal{B}}$, respectively. Here the right $\mathcal{B}$-module structure on $\Phi_{\mathcal{B}}(\Gamma) = \Gamma/\mathcal{B}^+\Gamma$ comes from the right $\mathcal{B}$-module structure of Γ. The right $\mathcal{B}$-module structure on $\Psi_B(V) = \mathcal{A}\square_{\overleftarrow{\mathcal{A}}} V$ is given by $(\sum_i p_i \otimes q_i)b = \sum_i p_i b_{(-1)} \otimes q_i b_{(0)}$.

Corollary 6.3. *The functors $\Phi_{\mathcal{B}}$ and $\Psi_{\mathcal{B}}$ are mutually inverse equivalences of categories.*

6.2. Locally finite duals of $U_q(\mathfrak{g})$-modules induced by $U_q(\mathfrak{l}_S)$-modules

For $\lambda \in P_S^+$ define

$$\Omega(\lambda) := \{f \in (W^{M(\lambda)})^* \mid \dim(f U_q(\mathfrak{g})) < \infty\}.$$

Here the dual vector space $(W^{M(\lambda)})^*$ of the left $U_q(\mathfrak{g})$-module $W^{M(\lambda)}$ is endowed with a right $U_q(\mathfrak{g})$-module structure in the usual way by $(fu)(v) := f(uv)$ for all $f \in (W^{M(\lambda)})^*$, $u \in U_q(\mathfrak{g})$, $v \in W^{M(\lambda)}$. One has a canonical inclusion

$$c : \Omega(\lambda) \to U_q(\mathfrak{g})^*, \quad f \mapsto c_f := (u \mapsto f(u \otimes v_\lambda)).$$

We will freely use the inclusion c to consider $\Omega(\lambda)$ as a subset of $U_q(\mathfrak{g})^*$.

Lemma 6.4. *For all $\lambda \in P_S^+$ one has $\Omega(\lambda) \subset \mathcal{A}$. In particular one has $\Omega(0) = \mathcal{B}$. Moreover, $\Omega(\lambda)$ is a left $\mathcal{A}$-covariant $\mathcal{B}$-bimodule.*

Proof. The dual Hopf algebra $U_q(\mathfrak{g})^\circ$ of $U_q(\mathfrak{g})$ satisfies

$$U_q(\mathfrak{g})^\circ = \{a \in U_q(\mathfrak{g})^* \mid \dim(aU_q(\mathfrak{g})) < \infty\}. \tag{39}$$

Thus by definition $\Omega(\lambda) \subset U_q(\mathfrak{g})^\circ$. Moreover, $U_q(\mathfrak{g})^\circ$ contains $\mathcal{A}$ as the linear span of the matrix coefficients of the representations $V(\mu)$, $\mu \in P^+$. Recall that $U_q(\mathfrak{g})$ is semisimple and any irreducible finite dimensional representation of $U_q(\mathfrak{g})$ can be obtained by tensoring some $V(\mu)$ with a one-dimensional representation D_ν, $\nu \in \{-1, 1\}^r$, given by $K_i v = \nu_i v$ for all $v \in D_\nu$. As $\lambda \in P_S^+$ the finite dimensional $U_q(\mathfrak{g})$-module generated by c_f for $f \in \Omega(\lambda)$ decomposes into a direct sum of irreducible representations isomorphic to $V(\mu)$, $\mu \in P_+$. Thus one gets $\Omega(\lambda) \subset \mathcal{A}$.

Note that $\overline{U} := W^{M(0)}$ is a left $U_q(\mathfrak{g})$-module coalgebra. Let $^- : U_q(\mathfrak{g}) \to \overline{U}$ denote the canonical projection $u \mapsto u \otimes v_0$. Note that $W^{M(\lambda)}$ is a right and left $\overline{U}$-comodule, where the coaction is given by

$$\Delta_L(u \otimes v_\lambda) = \overline{u}_{(1)} \otimes u_{(2)} \otimes v_\lambda \in \overline{U} \otimes W^{M(\lambda)},$$
$$\Delta_R(u \otimes v_\lambda) = u_{(1)} \otimes v_\lambda \otimes \overline{u}_{(2)} \in W^{M(\lambda)} \otimes \overline{U}.$$

These coactions are compatible with each other and with the $U_q(\mathfrak{g})$-module structure of $W^{M(\lambda)}$. They induce the desired $\mathcal{B}$-bimodule structure on $\Omega(\lambda)$. ■

The above lemma implies in particular $\Omega(\lambda) \in {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$. Thus one can apply the functor Φ from the previous subsection. The following proposition states that up to dualization Ω is the inverse of $\Xi \circ \Phi$.

Proposition 6.5. *For all $\lambda \in P_S^+$ one has*

$$\Xi(\Phi(\Omega(\lambda))) = M(\lambda)^*.$$

Proof. Note first that by definition of the left $\mathcal{B}$-module structure of $\Omega(\lambda)$ one has $(\mathcal{B}^+\Omega(\lambda))(1 \otimes M(\lambda)) = 0$. Thus there exists a well defined pairing

$$\langle \cdot, \cdot \rangle_\lambda : \Omega(\lambda)/(\mathcal{B}^+\Omega(\lambda)) \times M(\lambda) \longrightarrow \mathbb{C}. \tag{40}$$

The pairing $\langle \cdot, \cdot \rangle_\lambda$ induces a map of right $U_q(\mathfrak{l}_S)$-modules

$$\varphi : \Xi(\Phi(\Omega(\lambda))) \to M(\lambda)^*.$$

As $\Omega(\lambda) \subset \mathcal{A}$ the induced map of quotients

$$i : \Omega(\lambda)/(\mathcal{B}^+\Omega(\lambda)) \to \mathcal{A}/\mathcal{B}^+\mathcal{A}$$

is also injective by Theorem 6.2. Moreover, let π denote the surjection

$$\pi : U_q(\mathfrak{l}_S) \to M(\lambda), \quad k \mapsto kv_\lambda.$$

Then the pairings (35) and (40) satisfy

$$\langle \overline{f}, \pi(k) \rangle_\lambda = \langle k, i(\overline{f}) \rangle$$

for all $\overline{f} \in \Omega(\lambda)/(\mathcal{B}^+\Omega(\lambda))$, $k \in U_q(\mathfrak{l}_S)$. As the pairing (35) is nondegenerate and i is injective this implies that φ is injective. By Theorem 6.2 as $M(\lambda)$ is irreducible it remains to show that $\Omega(\lambda) \neq 0$. To this end apply Lemma 2.1 to pick $\mu \in P^+$ such that there exists an embedding $M(\lambda) \hookrightarrow V(\mu)$ of $U_q(\mathfrak{l}_S)$-modules. Let v denote the image of v_λ under this embedding. Pick $g \in V(\mu)^*$ such that $g(v) \neq 0$ and let $c_{g,v} \in \mathbb{C}_q[G]$ denote the corresponding matrix coefficient. Then there is an element $f \in \Omega(\lambda) \setminus \{0\}$ defined by $f(u \otimes v_\lambda) = c_{g,v}(u)$. ■

7. *q*-Differential forms as locally finite duals

From now on we restrict to the case of irreducible flag manifolds G/P_S. Thus $S = \pi \setminus \{\alpha_s\}$ where α_s occurs in each positive root of $\mathfrak{g}$ with multiplicity at most one. Let again $\mathcal{B} \subset \mathbb{C}_q[G]$ be the left coideal subalgebra defined by (34).

7.1. q-Differential forms for irreducible flag manifolds

The aim of this section is to recall the structure of the canonical differential graded algebra over $\mathcal{B}$ constructed and investigated in [7,6].

To this end recall that a first order differential calculus (FODC) over $\mathcal{B}$ is a $\mathcal{B}$-bimodule Γ together with a $\mathbb{C}$-linear map

$$\mathrm{d} : \mathcal{B} \to \Gamma$$

such that $\Gamma = \mathrm{Lin}_\mathbb{C}\{a\,\mathrm{d}b\,c \mid a, b, c \in \mathcal{B}\}$ and d satisfies the Leibniz rule

$$\mathrm{d}(ab) = a\mathrm{d}b + \mathrm{d}ab.$$

If Γ possesses the structure of a left $\mathcal{A}$-comodule

$$\Delta_\Gamma : \Gamma \to \mathcal{A} \otimes \Gamma$$

such that

$$\Delta_\Gamma(a\mathrm{d}bc) = (\Delta_\mathcal{B} a)((\mathrm{Id} \otimes \mathrm{d})\Delta_\mathcal{B} b)(\Delta_\mathcal{B} c)$$

then Γ is called (left) covariant. A covariant FODC $\Gamma \neq \{0\}$ over $\mathcal{B}$ is called irreducible if it does not possess any nontrivial quotient (by a left covariant $\mathcal{B}$-bimodule). The dimension of a covariant FODC $\Gamma \neq \{0\}$ over $\mathcal{B}$ is defined by $\dim \Gamma = \dim_\mathbb{C} \Gamma/\mathcal{B}^+\Gamma$. Any finite dimensional covariant FODC over $\mathcal{B}$ is uniquely determined by its so called quantum tangent space

$$T_\Gamma = \{f \in \Gamma^* | f|_{\mathcal{B}^+\Gamma} = 0\},$$

(see [5, Lemma 6], [7, Remark 2.4]). The quantum tangent space can be considered as a subset of the dual coalgebra $\mathcal{B}^\circ$ of $\mathcal{B}$ via the map $f \mapsto (b \mapsto f(\mathrm{d}b))$. It is one of the main results of [6] that there exist precisely two finite dimensional irreducible covariant FODC $(\Gamma_\partial, \partial)$ and $(\Gamma_{\overline{\partial}}, \overline{\partial})$ over $\mathcal{B}$. The quantum tangent spaces of the FODC Γ_∂ and $\Gamma_{\overline{\partial}}$ [7, Propositions 3.3, 3.4] are given by

$$T_\partial = (\mathrm{ad}\ U_q(\mathfrak{l}_S))F_s, \qquad T_{\overline{\partial}} = (\mathrm{ad}\ U_q(\mathfrak{l}_S))E_s, \tag{41}$$

respectively, considered as subspaces of $\mathcal{B}^\circ$ via the pairing (12). Moreover, the FODC Γ_∂ and $\Gamma_{\overline{\partial}}$ satisfy

$$\mathcal{B}^+\Gamma_\partial = \Gamma_\partial\mathcal{B}^+, \qquad \mathcal{B}^+\Gamma_{\overline{\partial}} = \Gamma_{\overline{\partial}}\mathcal{B}^+. \tag{42}$$

The direct sum $\Gamma_\mathrm{d} = \Gamma_\partial \oplus \Gamma_{\overline{\partial}}$ with the map $\mathrm{d} = \partial \oplus \overline{\partial}$ is a covariant FODC which is a q-analogue of the Kähler differentials over the affine algebraic variety G/L_S.

Consult [7, Section 2.3.2] for the definition of the universal differential calculus of a FODC (Γ, d). Let $\Gamma^\wedge_{\partial,\mathrm{u}}$, $\Gamma^\wedge_{\overline{\partial},\mathrm{u}}$, and $\Gamma^\wedge_{\mathrm{d},\mathrm{u}}$ denote the universal differential calculi of the FODC $(\Gamma_\partial, \partial)$, $(\Gamma_{\overline{\partial}}, \overline{\partial})$, and $(\Gamma_\mathrm{d}, \mathrm{d})$, respectively. The following theorem is contained in [7, Propositions 3.6, 3.7, 3.11].

Theorem 7.1. (i) *The multiplicity of weight spaces of the left $U_q(\mathfrak{l}_S)$-module $(\Xi \circ \Phi(\Gamma^{\wedge k}_{\partial,\mathrm{u}}))^* = (\Gamma^{\wedge k}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\partial,\mathrm{u}})^*$ coincides with the multiplicity of weight spaces of the left $U(\mathfrak{l}_S)$-module $\Lambda^k(\mathfrak{g}/\mathfrak{p}_S)$. In particular*

$$\dim_\mathbb{C}(\Gamma^{\wedge k}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\partial,\mathrm{u}}) = \binom{\dim_\mathbb{C}(\mathfrak{g}/\mathfrak{p}_S)}{k}.$$

(ii) *The multiplicity of weight spaces of the left $U_q(\mathfrak{l}_S)$-module $(\Xi \circ \Phi(\Gamma^{\wedge k}_{\overline{\partial},\mathrm{u}}))^* = (\Gamma^{\wedge k}_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\overline{\partial},\mathrm{u}})^*$ coincides with the multiplicity of weight spaces of the left $U(\mathfrak{l}_S)$-module $\Lambda^k(\mathfrak{g}/\mathfrak{p}_S)^*$. In particular*

$$\dim_\mathbb{C}(\Gamma^{\wedge k}_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\overline{\partial},\mathrm{u}}) = \binom{\dim_\mathbb{C}(\mathfrak{g}/\mathfrak{p}_S)}{k}.$$

(iii) *For all $k \in \mathbb{N}_0$ the canonical map*

$$\bigoplus_{i+j=k} \Gamma^{\wedge i}_{\partial,\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge i}_{\partial,\mathrm{u}} \otimes \Gamma^{\wedge j}_{\overline{\partial},\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge j}_{\overline{\partial},\mathrm{u}} \to \Gamma^{\wedge k}_{\mathrm{d},\mathrm{u}}/\mathcal{B}^+\Gamma^{\wedge k}_{\mathrm{d},\mathrm{u}} \tag{43}$$

is an isomorphism. In particular

$$\dim \Gamma^{\wedge k}_{\mathrm{d},\mathrm{u}} = \binom{2\dim_\mathbb{C}(\mathfrak{g}/\mathfrak{p}_S)}{k}.$$

The above theorem implies in particular, that $\Phi(\Gamma^{\wedge 2\dim(\mathfrak{g}/\mathfrak{p}_S)}_{\mathrm{d},\mathrm{u}})$ is the trivial one-dimensional left $\overleftarrow{\mathcal{A}}$-comodule. Moreover, by (42) the right $\mathcal{B}$-action on $\Phi(\Gamma^{\wedge 2\dim(\mathfrak{g}/\mathfrak{p}_S)}_{\mathrm{d},\mathrm{u}})$ is trivial, i.e. $\gamma b = \varepsilon(b)\gamma$ for all $b \in \mathcal{B}$, $\gamma \in \Phi(\Gamma^{\wedge 2\dim(\mathfrak{g}/\mathfrak{p}_S)}_{\mathrm{d},\mathrm{u}})$. Hence by the categorical equivalence in Corollary 6.3 the covariant $\mathcal{B}$-bimodules $\Gamma^{\wedge 2\dim(\mathfrak{g}/\mathfrak{p}_S)}_{\mathrm{d},\mathrm{u}}$ and $\mathcal{B}$ are isomorphic. This observation implies the following corollary.

Corollary 7.2. $\Gamma^{\wedge 2\dim(\mathfrak{g}/\mathfrak{p}_S)}_{\mathrm{d},\mathrm{u}}$ *is a free left and right $\mathcal{B}$-module generated by one left $\mathbb{C}_q[G]$-coinvariant element $\omega_{vol} \in \Gamma^{\wedge 2\dim(\mathfrak{g}/\mathfrak{p}_S)}_{\mathrm{d},\mathrm{u}}$ satisfying $\omega_{vol}\, b = b\, \omega_{vol}$ for all $b \in \mathcal{B}$.*

7.2. The differential calculus $\Gamma^\wedge_{\partial,\mathrm{u}}$

One is now in a position to construct the differential graded algebras $\Gamma^\wedge_{\partial,\mathrm{u}}$, $\Gamma^\wedge_{\overline{\partial},\mathrm{u}}$, and $\Gamma^\wedge_{\mathrm{d},\mathrm{u}}$ from [7] as locally finite duals of BGG-like sequences of $U_q(\mathfrak{g})$-modules induced by $U_q(\mathfrak{l}_S)$-modules. We begin with $\Gamma^\wedge_{\partial,\mathrm{u}}$. Consider the BGG resolution

$$C^S_{*,0}: \qquad 0 \longrightarrow C^S_{\dim\mathfrak{g}/\mathfrak{p}_S,0} \xrightarrow{\varphi^S_{\dim(\mathfrak{g}/\mathfrak{p}_S)}} \cdots \xrightarrow{\varphi^S_1} C^S_{0,0} \xrightarrow{\varepsilon_\mu} V(0) \longrightarrow 0, \tag{44}$$

of the trivial $U_q(\mathfrak{g})$-module $V(0)$, the corresponding sequence (20) obtained by applying the functor _

$$\underline{C}^S_{*,0}:\ 0 \longrightarrow \underline{C}^S_{\dim\mathfrak{g}/\mathfrak{p}_S,0} \xrightarrow{\underline{\varphi}^S_{\dim(\mathfrak{g}/\mathfrak{p}_S)}} \cdots \xrightarrow{\underline{\varphi}^S_1} \underline{C}^S_{0,0} \xrightarrow{\varepsilon_\mu} U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}_S^{\mathrm{op}})} V(0) \longrightarrow 0,$$

and its locally finite dual

$$\Omega^{*,0}: \qquad 0 \longleftarrow \Omega^{\dim\mathfrak{g}/\mathfrak{p}_S,0} \xleftarrow{\partial_{\dim(\mathfrak{g}/\mathfrak{p}_S)}} \cdots \xleftarrow{\partial_1} \Omega^{0,0} \cong \mathcal{B} \longleftarrow \mathbb{C} \longleftarrow 0,$$

where

$$\Omega^{n,0} \cong \bigoplus_{w\in W^S,\ l(w)=n} \Omega(w.0).$$

Recall from Section 4.2 that the differentials of the complexes $C^S_{*,0}$ and $\underline{C}^S_{*,0}$ are given in terms of elements $y_{w,w'} \in U_q(\mathfrak{n}^-)$ where $w, w' \in W^S$ and $w \to w'$. In the case of irreducible flag manifolds the simple reflection s_s corresponding to α_s is the only element in W^S of length one. Note that $s_s.0 = -\alpha_s$. Thus the differential $\underline{\varphi}_1 : V^{M(-\alpha_s)} \to V^{M(0)}$ is determined by $y_{s_s,e} = F_s$ up to multiplication by a nonzero factor. The corresponding differential

$$\partial_1 : \mathcal{B} \cong \Omega^{0,0} \to \Omega(-\alpha_s) \cong \Omega^{1,0}$$

satisfies the Leibniz rule. Indeed, for all $a, b \in \mathcal{B}$, $u \in U_q(\mathfrak{g})$ one has

$$\begin{aligned}\partial_1(ab)(u \otimes v_{-\alpha_s}) &= (ab)(uF_s \otimes v_0)\\ &= a(u_{(1)}F_s \otimes v_0)b(u_{(2)} \otimes v_0) + a(u_{(1)}K_s^{-1} \otimes v_0)b(u_{(2)}F_s \otimes v_0)\\ &= (\partial_1(a)b + a\partial_1(b))(u \otimes v_{-\alpha_s}).\end{aligned}$$

Lemma 7.3. $(\partial_1 : \mathcal{B} \to \Omega^{1,0})$ *is a covariant FODC isomorphic to* $(\partial : \mathcal{B} \to \Gamma_\partial)$.

Proof. Recall from (41) that T_∂ is an irreducible $U_q(\mathfrak{l}_S)$-module of highest weight $-\alpha_s$. Taking duals one obtains that $M(-\alpha_s)^*$ is isomorphic to $\Gamma_\partial/\mathcal{B}^+\Gamma_\partial$ as a right $U_q(\mathfrak{l}_S)$-module. Proposition 6.5 and the categorical equivalence now imply that $\Omega^{1,0} \cong \Omega(-\alpha_s)$ is an $\mathcal{A}$-covariant $\mathcal{B}$-bimodule isomorphic to Γ_∂.

As $M(-\alpha_s)$ is an irreducible $U_q(\mathfrak{l}_S)$-module it remains to check that $\partial_1 \neq 0$. This is a special case of the following lemma which is proved independently of the above claim. ■

Lemma 7.4. *For all* $n \in \mathbb{N}_0$ *the map*

$$\psi_n : \mathcal{B} \otimes \Omega^{n,0} \to \Omega^{n+1,0}, \qquad b \otimes \omega \mapsto b\,\partial_{n+1}\omega$$

is surjective.

Proof. It suffices to show that for any $w, w' \in W^S$ such that $w \to w'$ one has

$$y_{w,w'} \notin U_q(\mathfrak{g})U_q(\mathfrak{l}_S)^+ \tag{45}$$

where $U_q(\mathfrak{l}_S)^+ = \ker\varepsilon \cap U_q(\mathfrak{l}_S)$ denotes the augmentation ideal of $U_q(\mathfrak{l}_S)$. Indeed, choose $f \in \Omega(w'.0)$ such that $f(1 \otimes v_{w'.0}) \neq 0$. Then for all $b \in \mathcal{B}$ one has

$$(\partial_{n+1}(bf))(1 \otimes v_{w.0}) = b(y_{w,w'(1)})f(y_{w,w'(2)} \otimes v_{w'.0}).$$

Since $\mathcal{B}$ separates $\overline{U} = U_q(\mathfrak{g})/U_q(\mathfrak{g})U_q(\mathfrak{l}_S)^+$ [6, Prop. 6.1] and $y_{w,w'} \notin U_q(\mathfrak{g})U_q(\mathfrak{l}_S)^+$ by Assumption (45) and $y_{w,w'} \in U_q(\mathfrak{n}^-)$, one can choose $b \in \mathcal{B}$ such that

$$b(y_{w,w'(1)})y_{w,w'(2)} = 1.$$

By assumption on f this implies

$$\partial_{n+1}(bf)|_{W^{M(w.0)}} \neq 0. \tag{46}$$

By the categorical equivalence and Proposition 6.5 the covariant $\mathcal{B}$-module $\Omega^{n+1,0}$ contains any irreducible covariant $\mathcal{B}$-submodule with multiplicity at most one. Since $\mathrm{Im}\,\psi_n$ is a covariant left $\mathcal{B}$-module relation (46) implies $\Omega(w.0) \subset \mathrm{Im}\psi_n$.

It remains to verify (45). Assume on the contrary that $y_{w,w'} \in V_- U_q(\mathfrak{l}_S^-)^+$ where $U_q(\mathfrak{l}_S^-)^+ = \ker \varepsilon \cap U_q(\mathfrak{l}_S^-)$. Then $y_{w,w'} \otimes v_{w'.0} \in V_- \otimes U_q(\mathfrak{l}_S^-)^+ v_{w'.0} \subset V^{M(w'.0)}$ is nonzero because by Remark 4.3 the standard map does not vanish. Since $(\mathrm{ad}\, U_q(\mathfrak{l}_S))V^- = V^-$ and $M(w'.0)$ is an irreducible $U_q(\mathfrak{l}_S)$-module there exists E_i, where $i \neq s$, such that $E_i y_{w,w'} \otimes v_{w'.0} \neq 0$. This is a contradiction to $h_{w,w'}(E_i \otimes v_{w.0}) = 0$. ∎

By Lemma 7.3 one has $\Omega^{1,0} = \mathrm{Lin}_{\mathbb{C}}\{a\,\partial_1 b \mid a, b \in \mathcal{B}\}$. Define a map

$$\begin{aligned} &\barwedge : \Omega^{1,0} \otimes_{\mathcal{B}} \Omega^{n,0} \to \Omega^{n+1,0} \\ &(a\,\partial_1 b) \otimes \omega \mapsto a\,\partial_1 b \barwedge \omega := a\,\partial_{n+1}(b\omega) - ab\,\partial_{n+1}\omega. \end{aligned} \tag{47}$$

Lemma 7.5. *The map $\barwedge$ is well defined.*

Proof. Recall from the last statement of Corollary 2.3 that $y_{w,w'} \in U_q(\mathfrak{n}^-)_{-\beta}$ where $\omega_s(\beta) = 1$. Therefore

$$(^-\otimes \mathrm{Id})\Delta y_{w,w'} - 1 \otimes y_{w,w'} \in \overline{U_q(\mathfrak{l}_S)F_s} \otimes U_q(\mathfrak{l}_S) \subset \overline{U} \otimes U_q(\mathfrak{g}) \tag{48}$$

where as before $\overline{U} = U_q(\mathfrak{g})/U_q(\mathfrak{g})U_q(\mathfrak{l}_S)^+$. To prove that $\barwedge$ is well defined consider $a_i, b_i \in \mathcal{B}$ such that $\sum_i a_i\,\partial_1 b_i = 0$ or equivalently

$$\sum_i a_i(u_{(1)}) b_i(u_{(2)} F_s) = 0 \quad \forall u \in U_q(\mathfrak{g}). \tag{49}$$

Since $a_i(ux) = a_i(u)\varepsilon(x)$ for all $u \in U_q(\mathfrak{g})$, $x \in U_q(\mathfrak{l}_S)$ formula (49) is equivalent to

$$\sum_i a_i(u_{(1)}) b_i(u_{(2)} x F_s) = 0 \quad \forall u \in U_q(\mathfrak{g}), x \in U_q(\mathfrak{l}_S). \tag{50}$$

Then for all $\omega \in \Omega^{n,0}$ and all $u \in U_q(\mathfrak{g})$ one has

$$\begin{aligned} &\sum_i (a_i \partial_{n+1}(b_i\omega) - a_i b_i \partial_{n+1}\omega)(u \otimes v_{w.0}) \\ &\quad = \sum_i a_i(u_{(1)}) \left[(b_i\omega)\left(\sum_{w'} u_{(2)} y_{w,w'} \otimes v_{w'.0}\right) - b_i(u_{(2)})\omega\left(\sum_{w'} u_{(3)} y_{w,w'} \otimes v_{w'.0}\right)\right] \\ &\quad = \sum_i a_i(u_{(1)}) \sum_{w'} b_i(u_{(2)} y^+_{w,w'(1)})\omega(u_{(3)} y_{w,w'(2)} \otimes v_{w'.0}) = 0 \end{aligned}$$

where $y^+ = y - \varepsilon(y)$ and the last equation follows from (48) and (50). Thus $\barwedge : \Omega^{1,0} \times \Omega^{n,0} \to \Omega^{n+1,0}$ is well defined. Moreover, by definition for $a, b \in \mathcal{B}$ and $\omega \in \Omega^{n,0}$ one has

$$((\partial_1 b)a) \barwedge \omega = (\partial_1(ba) - b\partial_1 a) \barwedge \omega = \partial_{n+1}(ba\omega) - b\partial_{n+1}(a\omega) = \partial_1 b \barwedge a\omega$$

and thus $\barwedge$ is defined on $\Omega^{1,0} \otimes_{\mathcal{B}} \Omega^{n,0}$. ∎

Lemma 7.6. *The map $\barwedge : \Omega^{1,0} \otimes_{\mathcal{B}} \Omega^{n,0} \to \Omega^{n+1,0}$ is surjective.*

Proof. One shows by induction that for all $a_1, \ldots, a_k \in \mathcal{B}$ the relation

$$\partial_1 a_1 \barwedge (\partial_1 a_2 \barwedge (\cdots \barwedge \partial_1 a_k)\cdots) = \partial_k(a_1 \partial_1 a_2 \barwedge (\cdots \barwedge \partial_1 a_k)\cdots) \tag{51}$$

holds. The claim of the lemma holds for $n = 0$. Using Lemma 7.4 and (51) one shows by induction on n that

$$\Omega^{n,0} = \mathrm{Lin}_{\mathbb{C}}\{a_0\partial_1 a_1 \barwedge (\partial_1 a_2 \barwedge (\cdots (\barwedge \partial_1 a_n)\cdots)) \mid a_0, a_1, \ldots, a_n \in \mathcal{B}\}. \quad ∎$$

As in [7] let $\Gamma^{\wedge}_{\partial,\mathrm{u}}$ denote the universal differential calculus with FODC $(\partial : \mathcal{B} \to \Gamma_\partial)$. Define a map

$$\iota^n : \Gamma^{\wedge n}_{\partial,\mathrm{u}} \to \Omega^{n,0}$$
$$a_0 \partial a_1 \wedge \partial a_2 \wedge \cdots \wedge \partial a_n \mapsto a_0 \partial_1 a_1 \,\overline{\wedge}\, (\partial_1 a_2 \,\overline{\wedge}\, (\cdots \overline{\wedge}\, \partial_1 a_n))$$

by repeated use of the map $\overline{\wedge}$.

Lemma 7.7. *The map ι^n is well defined.*

Proof. By definition of $\Gamma^{\wedge}_{\partial,\mathrm{u}}$ it suffices to show that for all $a_i, b_i \in \mathcal{B}$ such that $\sum_i a_i \partial b_i = 0$ and for all $\omega \in \Omega^{k,0}$, where $0 \le k \le n-2$, one has $\sum_i \partial_1 a_i \,\overline{\wedge}\, (\partial_1 b_i \,\overline{\wedge}\, \omega) = 0$. This can be seen as follows.

$$\begin{aligned}\sum_i \partial_1 a_i \,\overline{\wedge}\, (\partial_1 b_i \,\overline{\wedge}\, \omega) &= \sum_i \partial_{k+2}(a_i \partial_1 b_i \,\overline{\wedge}\, \omega) - \sum_i a_i \partial_{k+2}(\partial_1 b_i \,\overline{\wedge}\, \omega)\\ &= -\sum_i a_i \partial_{k+2}(\partial_{k+1}(b_i\omega) - b_i \partial_{k+1}\omega)\\ &= \sum_i a_i \partial_1 b_i \,\overline{\wedge}\, \partial_{k+1}\omega = 0. \qquad \blacksquare\end{aligned}$$

Note that by construction ι^* is a morphism of complexes. Moreover, by Theorem 7.1(i) and Proposition 2.2 the dimensions of the covariant left $\mathcal{B}$-bimodules $\Gamma^{\wedge n}_{\partial,\mathrm{u}}$ and $\Omega^{n,0}$ coincide. As ι^n is a surjective map of covariant left $\mathcal{B}$-modules by Lemma 7.6 the categorical equivalence implies that ι^n is an isomorphism.

Proposition 7.8. *The map $\iota^* : \Gamma^{\wedge *}_{\partial,\mathrm{u}} \to \Omega^{*,0}$ is an isomorphism of complexes of covariant $\mathcal{B}$-bimodules.*

7.3. The differential calculus $\Gamma^{\wedge}_{\overline{\partial},\mathrm{u}}$

Recall from Section 7.1 that there exists a second irreducible covariant FODC $(\Gamma_{\overline{\partial}}, \overline{\partial})$ over $\mathcal{B}$. It follows from (41) that $T_{\overline{\partial}} \cong M(-\alpha_s)^*$. As in the case $\Gamma^{\wedge}_{\partial,\mathrm{u}}$ the universal differential calculus $\Gamma^{\wedge}_{\overline{\partial},\mathrm{u}}$ can be obtained as the locally finite dual of a sequence of $U_q(\mathfrak{g})$-modules induced by $U_q(\mathfrak{l}_S)$-modules. This can be seen using the involutive algebra automorphism coalgebra antiautomorphism $\eta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined in Section 3.1. The exact sequences $C^S_{*,0}$ and $\underline{C}^S_{*,0}$ from the previous subsection can be endowed with a new $U_q(\mathfrak{g})$-module structure via η. Using the isomorphism $V^{M(\lambda)}_\eta \cong V^{M(\lambda)^*}$ and $W^{M(\lambda)}_\eta \cong W^{M(\lambda)^*}$ one obtains complexes

$$C^S_{0,*} : 0 \longrightarrow C^S_{0,\dim \mathfrak{g}/\mathfrak{p}_S} \xrightarrow{\overline{\varphi}^S_{\dim(\mathfrak{g}/\mathfrak{p}_S)}} \cdots \xrightarrow{\overline{\varphi}^S_1} C^S_{0,0} \xrightarrow{\overline{\varepsilon}_\mu} V(0) \longrightarrow 0,$$

and

$$\underline{C}^S_{0,*} : 0 \longrightarrow \underline{C}^S_{0,\dim \mathfrak{g}/\mathfrak{p}_S} \xrightarrow{\underline{\overline{\varphi}}^S_{\dim(\mathfrak{g}/\mathfrak{p}_S)}} \cdots \xrightarrow{\underline{\overline{\varphi}}^S_1} \underline{C}^S_{0,0} \xrightarrow{\underline{\overline{\varepsilon}}_\mu} U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}_S)} V(0) \longrightarrow 0,$$

where

$$C^S_{0,n} = \bigoplus_{w \in W^S,\, l(w)=n} V^{M(w.0)^*}, \qquad \underline{C}^S_{0,n} = \bigoplus_{w \in W^S,\, l(w)=n} W^{M(w.0)^*}.$$

If $w, w' \in W^S$ and $w \to w'$ then the component of the differential which maps $V^{M(w.0)^*} \to V^{M(w'.0)^*}$ is given by

$$u \otimes \xi_{-w.0} \mapsto u x_{w,w'} \otimes \xi_{-w'.0}$$

where $x_{w,w'} = \eta(y_{w,w'})$. Taking locally finite duals one obtains a complex

$$\Omega^{0,*} : 0 \longleftarrow \Omega^{0,\dim \mathfrak{g}/\mathfrak{p}_S} \xleftarrow{\overline{\partial}_{\dim(\mathfrak{g}/\mathfrak{p}_S)}} \cdots \xleftarrow{\overline{\partial}_1} \Omega^{0,0} \cong \mathcal{B} \longleftarrow \mathbb{C} \longleftarrow 0.$$

To show that $\Omega^{0,*}$ is isomorphic as a complex of covariant $\mathcal{B}$-bimodules to the complex $\Gamma^{\wedge}_{\overline{\partial},\mathrm{u}}$ constructed in [7, 3.3.2] the arguments of the last subsection can be repeated.

7.4. The differential calculus $\Gamma^{\wedge}_{\mathrm{d,u}}$

Combining the constructions from the previous two subsections the q-analogue of the de Rham complex over G/L_S can also be realized as a locally finite dual of a sequence of $U_q(\mathfrak{g})$-modules induced by $U_q(\mathfrak{l}_S)$-modules. To this end define

$$\underline{C}^S_{n,m} := \bigoplus_{\substack{w_1,w_2\in W^S\\ l(w_1)=n,\ l(w_2)=m}} W(w_1.0, w_2.0). \tag{52}$$

Recall that for each $w_1, w_2 \in W^S$ the $U_q(\mathfrak{g})$-module $W(w_1.0, w_2.0)$ is a cyclic module generated by $1 \otimes (v_{w_1.0} \otimes \xi_{-w_2.0})$. If $w_1, w_1', w_2 \in W^S$ and $w_1 \to w_1'$ define a map

$$\underline{h}_{w_1,w_1';w_2} : W(w_1.0, w_2.0) \to W(w_1'.0, w_2.0),$$
$$u \otimes (v_{w_1.0} \otimes \xi_{-w_2.0}) \mapsto u y_{w_1,w_1'} \otimes (v_{w_1'.0} \otimes \xi_{-w_2.0}).$$

Similarly, if $w_1, w_2, w_2' \in W^S$ and $w_2 \to w_2'$ define a map

$$\underline{h}_{w_1;w_2,w_2'} : W(w_1.0, w_2.0) \to W(w_1.0, w_2'.0),$$
$$u \otimes (v_{w_1.0} \otimes \xi_{-w_2.0}) \mapsto u x_{w_2,w_2'} \otimes (v_{w_1.0} \otimes \xi_{-w_2'.0}).$$

Note that the symbol _ in the above definitions of $\underline{C}^S_{n,m}$, $\underline{h}_{w_1,w_1';w_2}$, and $\underline{h}_{w_1,w_1';w_2}$ is only a formal reminiscence of the functor _ from Section 5.2. No functorial properties will be used in the present section.

Lemma 7.9. *For all $w_0, w_1, w_2 \in W^S$ such that $w_1 \to w_2$ the maps $\underline{h}_{w_1,w_2;w_0}$ and $\underline{h}_{w_0;w_1,w_2}$ are well defined.*

Proof. Note that

$$U_q(\mathfrak{g})\mathrm{Ann}_{U_q(\mathfrak{l}_S)}(v_{w_1.0} \otimes \xi_{-w_0.0}) = U_q(\mathfrak{g})\mathrm{Ann}_{U_q(\mathfrak{l}_S)}(y_{w_1,w_2} \otimes (v_{w_2.0} \otimes \xi_{-w_0.0}))$$

because $M(w_1.0) \otimes M(w_0.0)^*$ and $U_q(\mathfrak{l}_S)y_{w_1,w_2} \otimes (v_{w_2.0} \otimes \xi_{-w_0.0}) \subset W(w_2.0, w_0.0)$ are isomorphic as $U_q(\mathfrak{l}_S)$-modules by Proposition 5.4. This proves that $\underline{h}_{w_1,w_2;w_0}$ is well defined. The second statement follows analogously. ■

For each $w_2 \in W^S$ one obtains a sequence

$$\underline{C}^S_{*,w_2} : \quad \cdots \xrightarrow{\underline{h}_{n+1,w_2}} \underline{C}^S_{n,w_2} \xrightarrow{\underline{h}_{n,w_2}} \underline{C}^S_{n-1,w_2} \xrightarrow{\underline{h}_{n-1,w_2}} \cdots \xrightarrow{\underline{h}_{1,w_2}} \underline{C}^S_{0,w_2} \tag{53}$$

where

$$\underline{C}^S_{n,w_2} = \bigoplus_{w_1\in W^S,\, l(w_1)=n} W(w_1.0, w_2.0), \qquad \underline{h}_{n,w_2} = \sum_{\substack{w_1,w_1'\in W^S,\ l(w_1)=n\\ w_1\to w_1'}} \underline{h}_{w_1,w_1';w_2}.$$

This sequence satisfies $\underline{h}_{n,w_2}\underline{h}_{n+1,w_2} = 0$ for all $n \in \mathbb{N}$. Indeed, the exactness of the sequence (20) implies that for any $w_1, w_1'' \in W^S$ such that $l(w_1) = n + 1$ and $l(w_1'') = n - 1$ one has

$$\sum_{\substack{w_1'\in W^S,\\ w_1\to w_1'\to w_1''}} y_{w_1,w_1'} y_{w_1',w_1''} \in U_q(\mathfrak{n}^-)\mathrm{Ann}_{U_q(\mathfrak{l}_S^-)}(v_{w_1''.0}).$$

Similarly, for each $w_1 \in W^S$ one has a complex

$$\underline{C}^S_{w_1,*} : \quad \cdots \xrightarrow{\underline{h}_{w_1,n+1}} \underline{C}^S_{w_1,n} \xrightarrow{\underline{h}_{w_1,n}} \underline{C}^S_{w_1,n-1} \xrightarrow{\underline{h}_{w_1,n-1}} \cdots \xrightarrow{\underline{h}_{w_1,1}} \underline{C}^S_{w_1,0} \tag{54}$$

where

$$\underline{C}^S_{w_1,n} = \bigoplus_{w_2\in W^S,\ l(w_2)=n} W(w_1.0, w_2.0), \qquad \underline{h}_{w_1,n} = \sum_{\substack{w_2,w_2'\in W^S,\ l(w_2)=n\\ w_2\to w_2'}} \underline{h}_{w_1;w_2,w_2'}.$$

To prove exactness of the sequences (53) and (54) we extend the filtrations defined in Section 5.3 to the vector spaces $\underline{C}^S_{n,w_2}$ and $\underline{C}^S_{w_1,n}$. Define

$$\mathcal{F}_2^k\underline{C}^S_{n,w_2} = \bigoplus_{\substack{w_1\in W^S,\\ l(w_1)=n}} \mathcal{F}_2^k W(w_1.0, w_2.0), \qquad \mathcal{F}_1^k\underline{C}^S_{w_1,n} = \bigoplus_{\substack{w_2\in W^S,\\ l(w_2)=n}} \mathcal{F}_1^k W(w_1.0, w_2.0).$$

Lemma 7.10. *For any $w_1, w_2 \in W^S$ the complexes $\underline{C}^S_{w_1,*}$ and $\underline{C}^S_{*,w_2}$ are filtered by the filtrations $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.*

Proof. Consider $w_1, w_1', w_2' \in W^S$ such that $w_1 \to w_1'$. We show that

$$\underline{h}_{w_1,w_1';w_2}(\mathcal{F}_2^k W(w_1.0, w_2.0)) \subset \mathcal{F}_2^k W(w_1'.0, w_2.0).$$

Define $\mathcal{F}^k M(w_2.0)^* = \bigoplus_{\mathrm{ht}(\mu+w_2.0)\le k} M(w_2.0)^*_\mu$. Then

$$\begin{aligned}\mathcal{F}_2^k W(w_1.0, w_2.0) &\subset V_+U_q(\mathfrak{n}^-)\otimes v_{w_1.0}\otimes \mathcal{F}^k M(w_2.0)^*\\ &= \sum_{\mathrm{ht}(\beta)\le k} V_+U_q(\mathfrak{n}^-)U_q(\mathfrak{l}_S^+)_\beta \otimes (v_{w_1.0}\otimes \xi_{-w_2.0}).\end{aligned}$$

Therefore

$$\begin{aligned}\underline{h}_{w_1,w_1';w_2}(\mathcal{F}_2^k W(w_1.0, w_2.0)) &\subset \sum_{\mathrm{ht}(\beta)\le k} V_+U_q(\mathfrak{n}^-)U_q(\mathfrak{l}_S^+)_\beta y_{w_1,w_1'}\otimes (v_{w_1'.0}\otimes \xi_{-w_2.0})\\ &\subset V_+U_q(\mathfrak{n}^-)\otimes (v_{w_1'.0}\otimes \mathcal{F}^k M(w_2.0)^*)\\ &\subset V_+V_-\otimes M(w_1'.0)\otimes \mathcal{F}^k M(w_2.0)^*.\end{aligned}$$

The statement about $\underline{C}^S_{w_1,*}$ and $\mathcal{F}_1$ is verified analogously. ■

Let $\mathrm{gr}_{\mathcal{F}_2}\underline{C}^S_{*,w_2}$ and $\mathrm{gr}_{\mathcal{F}_1}\underline{C}^S_{w_1,*}$ denote the graded complexes associated to the filtrations $\mathcal{F}_2$ and $\mathcal{F}_1$, respectively.

Lemma 7.11. *One has isomorphisms of complexes*

$$\mathrm{gr}_{\mathcal{F}_2}\underline{C}^S_{*,w_2} \cong \underline{C}^S_{*,0}\otimes M(w_2.0)^* \tag{55}$$

$$\mathrm{gr}_{\mathcal{F}_1}\underline{C}^S_{w_1,*} \cong \underline{C}^S_{0,*}\otimes M(w_1.0) \tag{56}$$

for $*\ge 0$.

Proof. For $e \in \sum_{\mathrm{ht}(\beta)\le k} U_q(\mathfrak{l}_S^+)_\beta$ and $u \in U_q(\mathfrak{g})$ one obtains in analogy to the proof of Lemma 7.10

$$\begin{aligned}h_{w_1,w_1';w_2}(u\otimes (v_{w_1.0}\otimes e\xi_{-w_2.0})) &= uey_{w_1,w_1'}\otimes (v_{w_1.0}\otimes \xi_{-w_2.0})\\ &\in uy_{w_1,w_1'}\otimes (v_{w_1.0}\otimes e\xi_{-w_2.0}) + \mathcal{F}_2^{k-1}W(w_1'.0, w_2.0).\end{aligned}$$

This shows (55) and (56) is verified analogously. ■

Proposition 7.12. *For any $w_1, w_2 \in W^S$ and $n \in \mathbb{N}$ the complexes $\underline{C}^S_{w_1,*}$ and $\underline{C}^S_{*,w_2}$ are exact in $\underline{C}^S_{w_1,n}$ and $\underline{C}^S_{n,w_2}$, respectively.*

Proof. This follows immediately from Lemma 7.11 and the exactness of the complexes $\underline{C}^S_{*,0}$ and $\underline{C}^S_{0,*}$. ■

The above proposition is one main step in order to prove that $\underline{C}^S_{*,*}$ defined in (52) together with the maps

$$\underline{h}_{n,*}: \underline{C}^S_{n,*}\to \underline{C}^S_{n-1,*}, \qquad \underline{h}_{n,m} := \sum_{w_2\in W^S,\, l(w_2)=m} \underline{h}_{n,w_2}$$

$$\overline{\underline{h}}_{*,m}: \underline{C}^S_{*,m}\to \underline{C}^S_{*,m-1}, \qquad \overline{\underline{h}}_{n,m} := (-1)^n \sum_{w_1\in W^S,\, l(w_1)=n} \underline{h}_{w_1,m}$$

is a double complex.

Proposition 7.13. $(\underline{C}^S_{*,*}, \underline{h}_{*,*}, \overline{h}_{*,*})$ *is a double complex, i.e. for any* $n, m \in \mathbb{N}$ *the relation*

$$\overline{h}_{n-1,m} \circ \underline{h}_{n,m} + \underline{h}_{n,m-1} \circ \overline{h}_{n,m} = 0 \tag{57}$$

holds.

Proof. Note first that the claim of the proposition holds for $n = m = 1$. Indeed, recall that the simple reflection s_s corresponding to α_s is the only element in W^S of length one and that $s_s.0 = -\alpha_s$. Thus

$$\underline{C}^S_{0,1} = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(0) \otimes M(-\alpha_s)^*)$$
$$\underline{C}^S_{1,0} = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(-\alpha_s) \otimes M(0)^*)$$
$$\underline{C}^S_{1,1} = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(-\alpha_s) \otimes M(-\alpha_s)^*)$$

and up to a sign the maps $\underline{h}_{1,*}$ and $\overline{h}_{*,1}$ are determined by $y_{s_s,1} = F_s$ and $x_{s_s,1} = E_s$, respectively. Therefore

$$(\overline{h}_{0,1} \circ \underline{h}_{1,1} + \underline{h}_{1,0} \circ \overline{h}_{1,1})(u \otimes (v_{-\alpha_s} \otimes \xi_{\alpha_s})) = u(F_s E_s - E_s F_s) \otimes (v_0 \otimes \xi_0) = 0$$

for all $u \in U_q(\mathfrak{g})$.

Now the proof is performed by induction over n and m. Assume that (57) holds for some $n, m \in \mathbb{N}$. We will show that this implies (57) with n replaced by $n + 1$. The induction over m is performed analogously.

Note that (57) is equivalent to

$$0 = [y_{w_1,w_1'}, x_{w_2,w_2'}] \otimes (v_{w_1'.0} \otimes \xi_{-w_2'.0}) \in \underline{C}^S_{n-1,m-1}$$

for all $w_1, w_1', w_2, w_2' \in W^S$ such that $l(w_1) = n, l(w_2) = m$ and $w_1 \to w_1', w_2 \to w_2'$. In particular one gets for any $w_1'' \in W^S$ such that $l(w_1'') = n + 1$ the relation

$$\sum_{\substack{w_1 \in W^S, l(w_1)=n \\ w_1'' \to w_1 \to w_1'}} y_{w_1'',w_1} [y_{w_1,w_1'}, x_{w_2,w_2'}] \otimes (v_{w_1'.0} \otimes \xi_{-w_2'.0}) = 0$$

and thus using $\underline{h}_{n,*} \circ \underline{h}_{n+1,*} = 0$ one obtains

$$\sum_{\substack{w_1 \in W^S, l(w_1)=n \\ w_1'' \to w_1 \to w_1'}} [y_{w_1'',w_1}, x_{w_2,w_2'}] y_{w_1,w_1'} \otimes (v_{w_1'.0} \otimes \xi_{-w_2'.0}) = 0.$$

By the exactness stated in Proposition 7.12 for all $w_1'', w_1, w_2, w_2' \in W^S, w_1'' \to w_1, w_2 \to w_2', l(w_1) = n, l(w_2) = m$ there exist elements $u_{w_1'''} \in U_q(\mathfrak{g})$ such that the relation

$$[y_{w_1'',w_1}, x_{w_2,w_2'}] \otimes (v_{w_1.0} \otimes \xi_{-w_2'.0}) = \sum_{\substack{w_1''' \in W^S \\ w_1''' \to w_1}} u_{w_1'''} y_{w_1''',w_1} \otimes (v_{w_1.0} \otimes \xi_{-w_2'.0}) \tag{58}$$

holds. By Corollary 5.6 the above relation implies

$$0 = [y_{w_1'',w_1}, x_{w_2,w_2'}] \otimes (v_{w_1.0} \otimes \xi_{-w_2'.0}) \in \underline{C}^S_{n,m-1}.$$

This is relation (57) with n replaced by $n + 1$. ■

Using Proposition 7.13 one can now define a double complex of covariant $\mathcal{B}$-bimodules setting

$$\Omega^{m,n} = \bigoplus_{\substack{w_1,w_2 \in W^S \\ l(w_1)=m, l(w_2)=n}} \Omega(w_1, w_2)$$

where for $w_1, w_2 \in W^S$ we define

$$\Omega(w_1, w_2) := \{f \in W(w_1.0, w_2.0)^* \mid \dim(f U_q(\mathfrak{g})) < \infty\}.$$

Note that by definition $\Omega^{m,n} = \{f \in C_{m,n} \mid \dim(fU_q(\mathfrak{g})) < \infty\}$. Thus the differentials $\underline{h}_{*,*}$ and $\overline{\underline{h}}_{*,*}$ on $C_{*,*}$ induce differentials $\partial^{*,*}$ and $\overline{\partial}^{*,*}$ on $\Omega^{*,*}$, respectively. Proposition 7.13 implies that $(\Omega^{*,*}, \partial^{*,*}, \overline{\partial}^{*,*})$ is a double complex. Let (Ω^*, d^*) denote the corresponding total complex, i. e. $\Omega^n = \bigoplus_{k+l=n} \Omega^{k,l}$ and the differential $\mathrm{d}^n : \Omega^{n-1} \to \Omega^n$ is given by

$$\mathrm{d}^n = \sum_{m=0}^{n} \partial^{n-m,m} + \overline{\partial}^{m,n-m}.$$

We are now in a position to formulate the main result of this paper.

Theorem 7.14. *There exists an isomorphism $\iota^* : \Gamma^{\wedge}_{\mathrm{d,u}} \to \Omega^*$ of complexes of covariant $\mathcal{B}$-bimodules.*

The proof is performed as in Section 7.2 up to slight modifications. The details are given for the convenience of the reader.

Note first that $(\mathrm{d}^1 : \mathcal{B} \to \Omega^1)$ is a covariant FODC isomorphic to the FODC $(\mathrm{d} : \mathcal{B} \to \Gamma^{\wedge 1}_{\mathrm{d,u}})$ constructed in [7]. Indeed, $\Omega^1 = \Omega^{1,0} \oplus \Omega^{0,1}$ and $\Gamma^{\wedge 1}_{\mathrm{d,u}} = \Gamma^{\wedge 1}_{\partial,\mathrm{u}} \oplus \Gamma^{\wedge 1}_{\overline{\partial},\mathrm{u}}$ and the isomorphisms of calculi $(\Omega^{1,0}, \partial^{1,0}) = (\Gamma^{\wedge 1}_{\partial,\mathrm{u}}, \partial)$ and $(\Omega^{0,1}, \partial^{0,1}) = (\Gamma^{\wedge 1}_{\overline{\partial},\mathrm{u}}, \overline{\partial})$ have been noted in Sections 7.2 and 7.3, respectively.

Next note that the following analogue of Lemma 7.4 holds.

Lemma 7.15. *The map*

$$\phi_n : \mathcal{B} \otimes \Omega^n \to \Omega^{n+1}, \qquad b \otimes \omega \mapsto b\,\mathrm{d}^{n+1}\omega$$

is surjective.

Proof. Note that for all k, l the covariant $\mathcal{B}$-modules $\Omega^{k,l+1}$ and $\Omega^{k+1,l}$ have no irreducible component in common. Indeed,

$$(\Xi \circ \Phi)\Omega^{k,l+1} = \bigoplus_{\substack{w_1,w_2 \in W^S \\ l(w_1)=k,\, l(w_2)=l+1}} M(w_1.0)^* \otimes M(w_2.0)$$

lies in the eigenspace corresponding to the eigenvalue $q^{k-(l+1)}$ of the action of the element $\tau(\omega_s)$ in the simply connected quantized universal enveloping algebra $\check{U}_q(\mathfrak{g})$ [9, 3.2.9]. Thus to prove surjectivity of ϕ_n it is sufficient to show that the maps

$$\phi_{k,l} : \mathcal{B} \otimes \Omega^{k,l} \to \Omega^{k+1,l}, \qquad b \otimes \omega \mapsto b\partial\omega$$
$$\overline{\phi}_{k,l} : \mathcal{B} \otimes \Omega^{k,l} \to \Omega^{k,l+1}, \qquad b \otimes \omega \mapsto b\overline{\partial}\omega$$

are surjective.

Let $w_1, w_2 \in W^S$ such that $\Omega(w_1, w_2) \subset \Omega^{k+1,l}$. Choose $w_1' \in W^S$ such that $w_1 \to w_1'$. Let $\{\xi_i\}$ be a basis of weight vectors of $M(w_2.0)^*$. By the categorical equivalence there exist elements $f_i \in \Omega(w_1', w_2)$ such that

$$f_i(1 \otimes v_{w_1'.0} \otimes \xi_j) = \delta_{ij}.$$

Using the fact that $\mathcal{B}$ separates $\overline{U} = U_q(\mathfrak{g})/U_q(\mathfrak{g})U_q(\mathfrak{l}_S)^+$ [6, Proposition 7] and Corollary 2.3 one sees that there exists an element $b_{w_1} \in \mathcal{B}$ such that

$$b_{w_1}(y_{w,w_1'(1)})y_{w,w_1'(2)} = \delta_{w,w_1} \quad \text{for all } w \in W^S,\ w \to w_1'.$$

One obtains

$$\partial^{k,l}(b_{w_1}f_i)(1 \otimes v_{w.0} \otimes \xi_j) = (b_{w_1}f_i)(y_{w,w_1'} \otimes v_{w.0} \otimes \xi_j) = \delta_{w,w_1}\delta_{i,j}.$$

Let $\{v_i\}$ denote a weight basis of $M(w_1.0)$. Acting with elements of $U_q(\mathfrak{l}_S)$ on $\partial^{k,l}(b_{w_1}f_i)$ one obtains elements g_{i,j,w_1} such that $g_{i,j,w_1}(1 \otimes v_k \otimes \xi_l) = \delta_{i,k}\delta_{j,l}$ and

$$g_{i,j,w_1}|_{W(w,w_2)} = 0 \quad \text{for all } w \in W^S,\ w \neq w_1.$$

Thus for the covariant $\mathcal{B}$-bimodule $\Lambda = \mathrm{Im}(\phi_{k,l}|_{\mathcal{B}\otimes\Omega(w_1',w_2)})$ the pairing

$$\Lambda/\mathcal{B}^+\Lambda \times \bigoplus_{\substack{w_1\in W^S\\ w_1\to w_1'}} M(w_1,w_2) \to \mathbb{C}$$

is nondegenerate. By the categorical equivalence one obtains

$$\Lambda = \bigoplus_{\substack{w_1\in W^S\\ w_1\to w_1'}} \Omega(w_1,w_2)$$

which proves the surjectivity of $\phi_{k,l}$. The surjectivity of $\overline{\phi}_{k,l}$ is proved analogously. ■

The remaining steps to identify (Ω^*, d^*) with $(\Gamma^\wedge_{\mathrm{d,u}}, \mathrm{d})$ are now straightforward analogues of the Lemmas 7.5–7.7 and of Proposition 7.8. The proofs are omitted. Define a map

$$\begin{aligned}&\overline{\wedge} : \Omega^1 \otimes_{\mathcal{B}} \Omega^n \to \Omega^{n+1},\\ &(a\,\mathrm{d}^0 b)\otimes\omega \mapsto a\,\mathrm{d}^0 b\,\overline{\wedge}\,\omega := a\,\mathrm{d}^n(b\omega) - ab\,\mathrm{d}^n\omega.\end{aligned} \tag{59}$$

Lemma 7.16. *The map $\overline{\wedge}$ is well defined.*

Lemma 7.17. *The map $\overline{\wedge} : \Omega^1 \otimes_{\mathcal{B}} \Omega^n \to \Omega^{n+1}$ is surjective.*

Define a map

$$\begin{aligned}&\iota^n : \Gamma^{\wedge n}_{\mathrm{d,u}} \to \Omega^n\\ &a_0\mathrm{d}a_1\wedge\mathrm{d}a_2\wedge\cdots\wedge\mathrm{d}a_n \mapsto a_0\mathrm{d}^0 a_1\,\overline{\wedge}\,(\mathrm{d}^0 a_2\,\overline{\wedge}\,(\cdots\overline{\wedge}\,\mathrm{d}^0 a_n))\end{aligned}$$

by repeated use of the map $\overline{\wedge}$.

Lemma 7.18. *The map ι^k is well defined.*

Proposition 7.19. *The map $\iota^* : \Gamma^{\wedge *}_{\mathrm{d,u}} \to \Omega^*$ is an isomorphism of complexes of covariant $\mathcal{B}$-bimodules.*

Proof of Theorem 7.14. This proof is now performed in complete analogy to the proof of Proposition 7.8. First note that by construction $\iota^* : \Gamma^{\wedge *}_{\mathrm{d,u}} \to \Omega^*$ is a morphism of complexes. Moreover, by Theorem 7.1(iii), the definition of Ω^n, and Proposition 2.2 the dimensions of the covariant $\mathcal{B}$-modules $\Gamma^{\wedge n}_{\mathrm{d,u}}$ and Ω^k coincide. As ι^n is a surjection of covariant $\mathcal{B}$-modules by Lemma 7.17 the categorical equivalence implies that ι^n is an isomorphism. ■

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Appendix. Commonly used notation

Symbols defined in Section 2.1 in order of appearance:

$\mathfrak{g}$, r, $\mathfrak{h}$, R, π, α_i, R^+, R^-, $\mathfrak{n}^+$, $\mathfrak{n}^-$, $(\cdot,\cdot)$, Q, P, $\alpha_i^\vee$, ht, ω_i, (a_{ij}), P^+, Q^+, $V(\mu)$, $\Pi(V(\mu))$, G, S, Q_S, Q_S^+, $R_S^\pm$, P_S, P_S^{op}, $\mathfrak{p}_S$, $\mathfrak{p}_S^{\mathrm{op}}$, $\mathfrak{l}_S$, L_S, P_S^+, $M(\lambda)$, W, s_α, W_S, W^S, l, $w.\mu$, ρ, $w\to w'$, $w\le w'$.

Section 3.1:

q	element of $\mathbb{C}$, not a root of unity
$U_q(\mathfrak{g})$	quantized enveloping algebra of $\mathfrak{g}$
K_i, K_i^{-1}, E_i, F_i	generators of $U_q(\mathfrak{g})$
$\Delta, \kappa, \varepsilon$	coproduct, antipode, and counit of $U_q(\mathfrak{g})$
ad	left adjoint action
η	algebra isomorphism coalgebra antiautomorphism of $U_q(\mathfrak{g})$
$U_q(\mathfrak{n}^+)$	subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i \mid 1 \le i \le r\}$
$U_q(\mathfrak{n}^-)$	subalgebra of $U_q(\mathfrak{g})$ generated by $\{F_i \mid 1 \le i \le r\}$
U^0	subalgebra of $U_q(\mathfrak{g})$ generated by $\{K_i, K_i^{-1} \mid 1 \le i \le r\}$
G_+	subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i K_i^{-1} \mid 1 \le i \le r\}$
$V(\mu)$	irreducible left $U_q(\mathfrak{g})$-module of highest weight $\mu \in P^+$
V_η	$U_q(\mathfrak{g})$-module with action twisted by η
$V(\mu)^*$	right or left $U_q(\mathfrak{g})$-module dual to $V(\mu)$
$c^\mu_{f,v}$	matrix coefficient of $V(\mu)$
$C^{V(\mu)}$	space of matrix coefficients of $V(\mu)$
$\mathbb{C}_q[G]$	q-deformed coordinate ring of G

Section 3.2:

$U_q(\mathfrak{l}_S)$	subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_j^{\pm 1} \mid \alpha_i \in S, \alpha_j \in \pi\}$
V_-	subalgebra of $U_q(\mathfrak{g})$ generated by $\{(\mathrm{ad}\, k)F_i \mid k \in U_q(\mathfrak{l}_S), \alpha_i \notin S\}$
V_+	subalgebra of $U_q(\mathfrak{g})$ generated by $\{(\mathrm{ad}\, k)(E_i K_i^{-1}) \mid k \in U_q(\mathfrak{l}_S), \alpha_i \notin S\}$
$U_q(\mathfrak{l}_S^-)$	$U_q(\mathfrak{n}^-) \cap U_q(\mathfrak{l}_S)$
$U_q(\mathfrak{l}_S^+)$	$G_+ \cap U_q(\mathfrak{l}_S)$
$U_q(\mathfrak{p}_S)$	subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_i, K_i^{\pm 1}, F_j \mid \alpha_i \in \pi, \alpha_j \in S\}$

Section 4.1:

$M(\lambda)$	irreducible left $U_q(\mathfrak{l}_S)$-module of highest weight $\lambda \in P_S^+$
$V^{M(\lambda)}$	$U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{p}_S)} M(\lambda)$ for $\lambda \in P_S^+$
V^λ	$V^{M(\lambda)}$ for $S = \emptyset$
ρ_S	$\sum_{\alpha \in R_S^+} \alpha/2$

Section 4.2:

C_j^S	$\bigoplus_{w \in W^S,\, l(w)=j} V^{M(w.\mu)}$ for a fixed $\mu \in P^+$
φ_j^S	boundary operator of BGG resolution
$f_{w,w'}$	fixed embedding of Verma modules $V^{w.\mu} \to V^{w'.\mu}$ if $w \le w'$
$s(w_1, w_2)$	± 1
$h_{w,w'}$	standard map induced by $s(w, w') f_{w,w'}$
$y^\mu_{w,w'}$	element of $U_q(\mathfrak{n}^-)$ such that $f_{w,w'}(u \otimes v_{w'.\mu}) = u y^\mu_{w,w'} \otimes v_{w'.\mu}$
$y_{w,w'}$	$s(w, w') y^0_{w,w'}$

Section 5.1:

$W^{M(\lambda)}$	$U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} M(\lambda)$ for $\lambda \in P_S^+$
W^λ	$W^{M(\lambda)}$ for $S = \emptyset$

Section 5.2:

Φ_λ	canonical surjection $W^{M(\lambda)} \to V^{M(\lambda)}$
$\mathcal{V}$	category of finite direct sums of $V^{M(\lambda)}$, $\lambda \in P_S^+$
$\mathcal{W}$	category of finite direct sums of $W^{M(\lambda)}$, $\lambda \in P_S^+$
$\_ : \mathcal{V} \to \mathcal{W}$	functor defined above Proposition 5.3
$U_q(\mathfrak{p}_S^{\mathrm{op}})$	subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_j, K_i^{\pm 1}, F_i \mid \alpha_i \in \pi, \alpha_j \in S\}$

Section 5.3:

$x^\mu_{w,w'}$	$\eta(y^\mu_{w,w'})$
$\theta_1, \overline{\theta}_1, \theta_2, \overline{\theta}_2$	$U_q(\mathfrak{l}_S)$-module homomorphisms defined in and before Proposition 5.4
$W(\mu,\nu)$	$U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{l}_S)} (M(\mu) \otimes M(\nu)^*)$
$\mathcal{F}_1^*, \mathcal{F}_2^*$	Filtrations of $W(\mu,\nu)$ defined by (27) and (28)

Section 6.1:

$\mathcal{A}$	$\mathbb{C}_q[G]$
$\mathcal{B}$	$\{b \in \mathcal{A} \mid b_{(1)} b_{(2)}(k) = \varepsilon(k) b$ for all $k \in U_q(\mathfrak{l}_S)\}$
$\mathcal{B}^+$	$\{b \in \mathcal{B} \mid \varepsilon(b) = 0\}$
$\overleftarrow{\mathcal{A}}$	$\mathcal{A}/\mathcal{B}^+\mathcal{A}$
$\langle \cdot, \cdot \rangle$	canonical pairing $U_q(\mathfrak{l}_S) \times \overleftarrow{\mathcal{A}} \to \mathbb{C}$
${}^{\overleftarrow{\mathcal{A}}}\mathcal{M}$	category of finite dimensional left $\overleftarrow{\mathcal{A}}$-comodules
$\mathcal{M}_{U_q(\mathfrak{l}_S)}$	category of finite direct sums of modules of the form $M(\lambda)^*$, $\lambda \in P_S^+$
Ξ	functor ${}^{\overleftarrow{\mathcal{A}}}\mathcal{M} \to \mathcal{M}_{U_q(\mathfrak{l}_S)}$ defined by (36)
$P \square_C Q$	cotensor product of P and Q over coalgebra C
${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$	category of left $\mathcal{A}$-covariant left $\mathcal{B}$-modules
Φ	functor ${}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M} \to {}^{\overleftarrow{\mathcal{A}}}\mathcal{M}$ defined by (37)
Ψ	functor ${}^{\overleftarrow{\mathcal{A}}}\mathcal{M} \to {}^{\mathcal{A}}_{\mathcal{B}}\mathcal{M}$ defined by (38)

Section 6.2:

$\Omega(\lambda)$	$\{f \in (W^{M(\lambda)})^* \mid \dim(f U_q(\mathfrak{g})) < \infty\}$
c	canonical inclusion $\Omega(\lambda) \to U_q(\mathfrak{g})^*$

Section 7.1:

$(\Gamma_\partial, \partial), (\Gamma_{\overline{\partial}}, \overline{\partial})$	the two irreducible covariant FODC over $\mathcal{B}$
$T_\partial, T_{\overline{\partial}}$	quantum tangent space of Γ_∂ and $\Gamma_{\overline{\partial}}$, respectively
$(\Gamma_{\mathrm{d}}, \mathrm{d})$	$(\Gamma_\partial \oplus \Gamma_{\overline{\partial}}, \partial \oplus \overline{\partial})$
$\Gamma^\wedge_{\partial,\mathrm{u}}, \Gamma^\wedge_{\overline{\partial},\mathrm{u}}, \Gamma^\wedge_{\mathrm{d},\mathrm{u}}$	universal differential calculus of Γ_∂, $\Gamma_{\overline{\partial}}$, and Γ_{d}, respectively.

References

[1] R.J. Baston, M.G. Eastwood, The Penrose Transform, Oxford University Press, Oxford, 1989.
[2] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Differential operators on the base affine space and the study of $\mathfrak{g}$-modules, in: I.M. Gelfand (Ed.), Lie Groups and their Representations, Halsted Press, New York, 1975, pp. 21–64.
[3] L. Dąbrowski, A. Sitarz, Dirac operator on the standard Podleś quantum sphere, in: P. Hajac, W. Pusz (Eds.), Noncommutative Geometry and Quantum Groups, in: Banach Center Publications, vol. 61, Polish Academy of Sciences, 2003, pp. 49–58.
[4] I. Heckenberger, S. Kolb, On the Bernstein–Gelfand–Gelfand resolution for Kac–Moody algebras and quantized enveloping algebras, Transform. Groups (in press). math.QA/0605460.
[5] I. Heckenberger, S. Kolb, Differential calculus on quantum homogeneous spaces, Lett. Math. Phys. 63 (2003) 255–264.
[6] I. Heckenberger, S. Kolb, The locally finite part of the dual coalgebra of quantized irreducible flag manifolds, Proc. London Math. Soc. 89 (2) (2004) 457–484.
[7] I. Heckenberger, S. Kolb, De Rham complex for quantized irreducible flag manifolds, J. Algebra 305 (2006) 704–741.
[8] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, in: Pure and Applied Mathematics, no. 80, Academic Press, New York, 1978.
[9] A. Joseph, Quantum groups and their primitive ideals, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1995.
[10] A. Joseph, A completion of the quantized enveloping algebra of a Kac–Moody algebra, J. Algebra 214 (1999) 235–275.
[11] M.S. Kébé, Sur la classification des $\mathcal{O}$-algèbres quantiques, J. Algebra 212 (1999) 626–659.
[12] A. Klimyk, K. Schmüdgen, Quantum Groups and their Representations, Springer-Verlag, Heidelberg, 1997.
[13] S. Kumar, Kac–Moody groups, their flag varieties and representation theory, in: Progress in Mathematics, no. 204, Birkhäuser, Boston, 2002.
[14] J. Lepowsky, A generalization of the Bernstein-Gelfand-Gelfand resolution, J. Algebra 49 (1977) 496–511.
[15] S. Majid, Noncommutative Riemannian and spin geometry of the standard q-sphere, Comm. Math. Phys. 256 (2) (2005) 255–285.
[16] E.F. Müller, H.-J. Schneider, Quantum homogeneous spaces with faithfully flat module structure, Israel. J. Math. 111 (1999) 157–190.
[17] W. Neidhardt, Irreducible subquotients and imbeddings of Verma modules, Algebras Groups Geom. 1 (1984) 127–136.
[18] P. Podleś, The classification of differential structures on quantum 2-spheres, Comm. Math. Phys. 150 (1992) 167–180.
[19] A. Rocha-Caridi, N.R. Wallach, Projective modules over graded Lie algebras. I, Math. Z. 180 (1982) 151–177.
[20] K. Schmüdgen, E. Wagner, Dirac operator and twisted cyclic cocycle on the standard Podleś quantum sphere, J. Reine Angew. Math. 574 (2004) 219–235.
[21] S.D. Sinel'shchikov, A. Stolin, L.L. Vaksman, Differential calculi on quantum prehomogeneous vector spaces, J. Math. Phys. 48 (2007), doi:10.1063/1.2746131, 073514.
[22] S.D. Sinel'shchikov, L.L. Vaksman, On q-analogues of bounded symmetric domains and Dolbeault complexes, Math. Phys. Anal. Geom. 1 (1) (1998) 75–100.
[23] M. Takeuchi, Relative Hopf modules - equivalences and freeness criteria, J. Algebra 60 (1979) 452–471.
[24] S.L. Woronowicz, Differential calculus on compact matrix pseudogroups (quantum groups), Comm. Math. Phys. 122 (1) (1989) 125–170.